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Pricing barrier stock options with discrete dividends by approximating analytical formulae

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Deriving accurate analytical formulas for pricing stock options with discrete dividend payouts is a hard problem even for the simplest vanilla options. This is because the falls in the stock price process due to discrete dividend payouts will significantly increase the mathematical difficulty in pricing the option. On the other hand, much literature uses other dividend settings to simplify the difficulty, but these settings may produce inconsistent pricing results. This paper derives accurate approximating formulae for pricing a popular path-dependent option, the barrier stock option, with discrete dividend payouts. The fall in stock price due to dividend payout at an exdividend date \( t \) is approximated by an accumulated price decrement due to a continuous dividend yield up to time \( t \). Thus, the stock price process prior to time \( t \) and after time \( t \) can be separately modelled by two different lognormal-diffusive stock processes which help us to easily derive analytical pricing formulae. Numerical experiments suggest that our formulae provide more accurate and coherent pricing results than other approximation formulae. Our formulae are also robust under extreme cases, like the high volatility (of the stock price) case. Besides, our formulae also extend the applicability of the first-passage model (a type of structural credit risk model) to measure how the firm’s payout influences its financial status and the credit qualities of other outstanding debts.

Keywords: Barrier option; Derivative pricing; Discrete dividend; First-passage model

JEL Classification: G1, G13

1. Introduction

Developing a feasible option pricing model that captures the phenomena of financial markets is an important issue in the financial field. Black and Scholes (1973) derive option pricing formulae for non-dividend-paying stocks. To deal with the dividend payout problem, Merton (1973) extends the Black-Scholes formula by assuming that the stock pays a fixed continuous dividend yield. However, most dividend payments are paid discretely rather than continuously. Pricing the option on the stock that pays a fixed dividend discretely seems to be first investigated in Black (1975). In addition, Ehrhardt and Brigham (2009) also argue that most stocks pay stable dividends discretely to maintain the investors’ confidence. Although this discrete-payment setting might be more realistic than the continuous one, it gives rise to significant mathematical difficulty in pricing the options. This is because the underlying stock price process becomes much more complicated due to the jumps caused by the discrete payments.

Pricing stock options with discrete dividend payouts has drawn a lot of attention in the literature. Frishling (2002) shows that the underlying stock price processes are usually modelled in the following three different ways. Model 1 suggests that the stock price minus the present value of future dividends over the life of the option follows a lognormal diffusion process (see Roll 1977, Geske 1979). Model 2 suggests that the stock price plus the forward values of the dividends paid from today up to the option’s maturity follows a lognormal diffusion process (see Heath and Jarrow 1988, Musiela and Rutkowski 1997). Model 3 suggests that the stock price falls by the amount of the dividend paid at the exdividend date, and follows the lognormal diffusion process between two adjacent exdividend dates. For pricing vanilla options, Frishling (2002) argues that these three models are incompatible with each other and generate very different prices. In addition, Frishling (2002), Bender and Vorst (2001), and Bos and Vandermark (2002) argue that only Model 3 can reflect the reality and generate consistent option prices. Apart from the three aforementioned models, Chiras and Manaster (1978) suggest that the discrete dividends can be transformed into a fixed continuous dividend yield. The vanilla
stock option can then be analytically solved by the Merton pricing formula (see Merton 1973). However, Dai and Lyuu (2009) show that the pricing results of their approach can deviate significantly from those generated by Model 3. On the other hand, pricing vanilla options under Model 3 can be mathematically intractable since the downward jumps due to the dividend payouts cause the stock price process to no longer follow a lognormal diffusion process. Bender and Vorst (2001), Bos and Vandermark (2002), Vellekoop and Nieuwenhuis (2006), and Dai and Lyuu (2009) derive the approximated distribution of the stock price at the maturity date and derive approximating analytical pricing formulae. Besides, Dai (2009) constructs a numerical pricing approach, namely, the ‘stair tree’, that produces accurate pricing results by faithfully modelling the evolution of the stock price process with downward jumps.

Similarly, pricing barrier stock options with the discrete dividend payout with the aforementioned models other than Model 3 can produce unreasonable pricing results (see Frisingh 2002). A barrier option is a popular exotic option whose payoff depends on whether the stock price at the maturity date is above a certain predetermined price level called a barrier. Reiner and Rubinstein (1991) derive an analytical pricing formula for the barrier option on the stock that pays a fixed continuous dividend yield. In their model, the stock price follows a lognormal diffusion process, and the joint density of the extreme stock price over the option life and the stock price at the option maturity date can therefore be derived by taking advantage of the reflection principle and Girsanov’s theorem. Unfortunately, their approach can not be directly extended to price barrier stock options with discrete dividends. While Zvan et al. (2000) and Gaudenzi and Zanette (2009) develop numerical methods to price barrier options under Model 3, to our knowledge, no announced papers derive analytical pricing formulae for pricing barrier stock options with discrete dividend payouts.

The major contribution of this paper is to derive approximating analytical formulae under Model 3. The numerical experiments in section 5 suggest that our formulae produce much better pricing results. Besides, our option pricing formulae also extend the applicability of the first-passage model—a credit risk model that simulates the evolution of the firm value and that triggers the default event once the firm value reaches the so-called ‘default boundary’. Therefore, the firm’s equity can be treated as a barrier call option on the firm value, and other outstanding debts can also be evaluated by taking advantage of our approach. Much of the literature puts restrictions on selling the firm’s assets to finance the loan repayments or dividend payouts (see Geske 1977, Kim et al. 1993, Leland 1994, Longstaff and Schwartz 1995). However, the empirical studies in Billett et al. (2007) suggest that up to 64.5% of their debt issue samples have asset sale clauses; that is, most debts allow the issuing firm to sell its assets to finance the debt repayments. Eom et al. (2004) also argue that 29 out of 31 bonds in their sample have asset sale clauses. They claim that selling assets to finance the repayment of one bond would damage the values of other outstanding bonds. Therefore, evaluating the impact of the asset sale clause on the debt value is important.

Lando (2004) argues that dealing with the asset sale clause can be mathematically intractable since the jumps in the firm’s value due to discrete payouts make the firm’s value process complicated. Indeed, by substituting the issuing firm’s value process and the discrete debt repayments for the stock price process and the dividend payout in our pricing formulae, we obtain new formulae to evaluate the credit risk for the debts that have asset sale clauses. Our numerical experiments suggest that the pricing results of our formulae match the empirical finding in Linn and Stock (2005): When the junior debt matures prior to the senior unsecured debt, the security of the senior unsecured debt is threatened and the default spread (of senior debt) may increase. On the other hand, another model that limits the firm to maintaining a constant continuous payout ratio—which is widely adopted by much of the literature such as Kim et al. (1993) and Longstaff and Schwartz (1995)—might fail to capture this finding.

Our pricing formulae are derived based on a piecewise stock price process designed to approximate the stock price process under Model 3. The stock price \( S(t) \) at time \( t \) given that no dividend is paid out during the time interval \([0, t]\) follows the lognormal diffusion process:

\[
S(t) = S(0)e^{\mu t + \sigma W(t)},
\]

where \( \mu \equiv r - 0.5\sigma^2 \) denotes the annual risk-free interest rate, \( \sigma \) denotes the volatility, and \( W(t) \) denotes the standard Brownian motion. Under Model 3, the stock pays dividends \( c_1, c_2, c_3, \ldots \) at exdividend dates \( t_1, t_2, t_3, \ldots \), respectively, where \( t_1 < t_2 < t_3, \ldots \). At the exdividend date \( t_1 \), the stock price falls by the amount \( c_1 \) due to the dividend payout as suggested in Black (1975) and Zvan et al. (2000). For convenience, define the stock return at time \( t \) as \( \frac{S(t)}{S(t-1)} \). The process of the stock return prior to the exdividend date \( t_1 \) can be expressed by the drifting Brownian motion: \( \mu t + \sigma W(t) \) as described in equation (1). However, the stock price at any time \( t \) between the exdividend dates \( t_1 \) and \( t_2 \) is

\[
S(t) = \left( S(0)e^{\mu t_1 + \sigma W(t_1) - c_1} \right) e^{\mu(t-1) + \sigma(W(t)-W(t_1))},
\]

and the stock return is no longer a drifting Brownian motion. To make the pricing problem tractable, the amount \( c_1 \) by which the stock price falls at the exdividend date \( t_1 \) is approximated by the accumulated price decrement caused by a continuous dividend yield \( q_1 \) paid from time 0 to time \( t_1 \). That is,

\[
S(t_1) = S(0)e^{\mu t_1 + \sigma W(t_1) - q_1} = S(0)e^{(\mu-q_1)t_1 + \sigma W(t_1)}.
\]

Thus, we construct another lognormal diffusion process with a continuous payout rate \( q_1 \) paid from time 0 to time \( t_1 \) to approximate the stock price process between the time interval \([t_1, t_2]\) in equation (2) as follows:

\[
S(t) = S(t_1)e^{\mu(t-t_1) + \sigma(W(t)-W(t_1))} = S(t_1)e^{\mu t - q_1 t_1 + \sigma W(t)},
\]

where \( t \in [t_1, t_2] \). Since \( q_1 \) in equation (3) can be approximately solved by the first-order Taylor expansion as an affine function of \( W(t_1) \), the process of the stock return between the exdividend dates \( t_1 \) and \( t_2 \), \( \mu t - q_1 t_1 + \sigma W(t) \), can be approximated by another drifting Brownian motion. Let the option maturity \( T < t_2 \) for simplicity. The joint distribution of the extreme stock price over the time interval \([0, t_1] \cup (t_1, T)\) and the stock price at time \( t_1 \) \( (T) \) can be solved for by applying the reflection principle and Girsanov’s theorem to the drifting Brownian motion \( \mu t + \sigma W(t) \) (another drifting Brownian motion \( \mu t - q_1 t_1 + \sigma W(t) \)). The pricing formulae can then
be derived by applying the risk-neutral valuation method to these two joint distributions. Our approach can be extended to the multiple-dividend case by repeating the aforementioned steps to derive the approximated stock return process between any two adjacent exdividend dates.

The remainder of this paper is organized as follows. Section 2 introduces the required financial and mathematical background knowledge. Section 3 derives mathematical properties that are useful for later deriving the pricing formulae. Our approximation pricing formulae are then derived in section 4. The experimental results given in section 5 verify the accuracy of our pricing formulae and demonstrate how our approach extends the applicability of the first-passage model. Section 6 concludes the paper.

2. Preliminaries

2.1. Barrier options and the first passage model

Assume that a barrier stock option with a strike price $K$ is initiated at time 0 and matures at time $T$. The payoff of an up-and-out option at maturity is as follows:

$$\text{payoff} = \begin{cases} (\theta S(T) - \theta K)^+ & \text{if } S_{\text{max}} < B \\ 0 & \text{if } S_{\text{max}} \geq B \end{cases},$$

where $(A)^+$ denotes max$(A, 0)$, $S_{\text{max}}$ denotes the maximum underlying stock price between time 0 and time $T$, $B$ denotes the barrier and $\theta$ equals 1 for call options and $-1$ for put options. Similarly, the payoff of a down-and-out option at maturity is as follows:

$$\text{payoff} = \begin{cases} (\theta S(T) - \theta K)^+ & \text{if } S_{\text{min}} > B \\ 0 & \text{if } S_{\text{min}} \leq B \end{cases},$$

where $S_{\text{min}}$ denotes the minimum stock price between time 0 and time $T$. For simplicity, our paper will focus on an up-and-out call option and the extensions to other barrier options are straightforward.

The same mathematical settings can be used to model the first passage model by redefining the symbol $B$ as the default boundary, $T$ as the debt maturity, and $K$ as the debt repayment due at maturity. The firm value process $S(t)$ is assumed to follow equations (1)–(4), where $\sigma$ denotes the volatility of the firm value and $c_1$ denotes the loan repayment or dividend payout at time $t$. The firm defaults once its value falls below the default boundary prior to the maturity date or it can not meet the debt obligation at the maturity date. Thus, the equity value can be evaluated as a down-and-out call option on the firm value and each debt issued by the firm can be priced by treating it as a contingent claim on the firm value.

2.2. Pricing barrier stock options without discrete dividends

The payoff of an up-and-out call depends on whether the underlying stock price process has ever risen above the barrier over the life of this option. The stock price process has risen above the barrier during the time interval $[0, \tau]$ if and only if the maximum stock price during the time interval $[0, \tau]$ is greater than the barrier. The following theorem, derived from the reflection principle and Girsanov’s theorem (see Shreve 2007), can be applied to describe the joint density of the stock price at time $\tau$ and the maximum stock price during the time interval $[0, \tau]$.

**Theorem 2.1** Let $\tilde{W}(t) = \sigma t + W(t)$ be a Brownian motion with a drift term $\sigma t$ and $M(\tau) = \max_{0 \leq t \leq \tau} \tilde{W}(t)$ be its maximum value over a certain time interval $[0, \tau]$. The joint density function of $(M(\tau), \tilde{W}(\tau))$ is given by

$$f_{M(\tau), \tilde{W}(\tau)}(m, w) = \frac{\left(2(2m - w)e^{-\frac{1}{2}w^2 - \frac{1}{2}\tau - \frac{1}{2}(2m - w)^2} - 1\right)}{\sqrt{2\pi \tau}}\text{if } m \geq w \\text{otherwise.}$$

(5)

The set of points $(m, w)$ that make density values non-zero, also known as the support of a density, is illustrated in figure 1(a).

Reiner and Rubinstein (1991) derive analytical formulae for barrier stock options with discrete dividends by Theorem 2.1. We derive some lemmas that can be used to derive their pricing formulae. These lemmas can also be applied to derive our barrier stock option pricing formulae with discrete dividend payouts. Define the stock return in equation (1), $\mu t + \sigma W(t)$ as $W(t)$, where the drifted Brownian motion $\tilde{W}(t)$ is defined as $\mu t + \sigma W(t)$. Define the maximum value of the Brownian motion $M(t)$ as $\max_{0 \leq t \leq \tau} \tilde{W}(t)$.

The value of an up-and-out call option $C$ can be derived as follows:

$$C = e^{-\tau T} E \left( (S(T) - K)^+ 1_{S(T) < B} \right)$$

$$= e^{-\tau T} E \left( (S(0)e^{\sigma \tilde{W}(T)} - K) 1_{S(0)e^{\sigma \tilde{W}(T)} \geq K, S(0)e^{\sigma \tilde{W}(T)} < B} \right)$$

$$= e^{-\tau T} E \left( (S(0)e^{\sigma \tilde{W}(T)} - K) 1_{\tilde{W}(T) \geq k, \tilde{W}(T) < b} \right),$$

(6)

where $k$ and $b$ in equation (6) stand for $\frac{1}{\sigma} \ln \frac{K}{\max SB}$ and $\frac{1}{\sigma} \ln \frac{B}{\max SB}$, respectively. By substituting equation (5) into equation (6) with $\alpha = \frac{\sigma}{\mu}$, we have

$$C = \int_k^b \int_{-\infty}^\infty e^{-\tau T} (S(0)e^{\sigma w} - K) f_{M(\tau), \tilde{W}(\tau)}(m, w) dmdw$$

(7)

$$= \int_k^b \int_{w}^{\infty} e^{-\tau T} (S(0)e^{\sigma w} - K) \frac{2(2m - w)e^{-\frac{1}{2}w^2 - \frac{1}{2}\tau - \frac{1}{2}(2m - w)^2}}{\sqrt{2\pi T}}dmdw,$$

(8)

where the domain of integral in equation (7), i.e., $\infty < m < b$ and $k \leq w < \infty$, is the support of the indicator function in equation (6) as illustrated in figure 1(b). The domain of integral in equation (8) is the intersection of the support of the joint density function $f_{M(\tau), \tilde{W}(\tau)}(m, w)$ and the support of indicator function $1_{\tilde{W}(T) \geq k, \tilde{W}(T) < b}$ as illustrated in figure 1(c).

In the double integral formula equation (8), since only $f_{M(\tau), \tilde{W}(\tau)}(m, w)$ contains the variable $m$, $\int w^b f_{M(\tau), \tilde{W}(\tau)}(m, w) dm$ can be evaluated first by the following lemma†:

†Proofs of this lemma is available upon request.
3. Derivations of useful mathematical properties

3.1. Approximate the stock price process under Model 3 piecewisely with lognormal diffusion processes

We derive a systematic approach for constructing a series of lognormal diffusion processes to piecewisely approximate the stock price process under Model 3. To be precise, we decompose the stock price process into several parts by ex-dividend dates. Each part of the stock price process is approximated by a lognormal diffusion process that makes the stock return process for this part of the stock price process a drifted Brownian motion. Therefore, Theorem 2.1 can be applied to derive the pricing formulae.

Note that the stock return process for the time interval $[0, t_1]$ is already a drifted Brownian motion, $\mu t + \sigma W(t)$, as illustrated in equation (1). So we do not need to derive the approximated process for this interval. On the other hand, the stock return process between the time interval $[t_1, t_2]$ (see equation (2)) is not a drifted Brownian motion due to the discrete dividend $c_1$ paid at time $t_1$. The stock price drop due to the dividend payout is approximated by the accumulated price decrement caused by a continuous dividend $q_1$ paid from time 0 to time $t_1$ as illustrated in equation (3), so the resulting stock price process after time $t_1$ can be expressed as equation (4). To make this modified price process a lognormal diffusion one, $q_1$ is approximately solved as a linear function of $W(t_1)$ from equation (3) as follows:

$$ S(0)e^{\mu t_1 + \sigma W(t_1)} - c_1 \approx S(0)e^{\mu t_1 + \sigma W(t_1)}(1 - q_1 t_1) $$

$$ q_1 \approx \frac{c_1 e^{-\mu t_1} (1 - \sigma W(t_1))}{t_1 S(0)}, $$(11)

where the first-order Taylor expansion $e^x \approx 1 + x$ is used in equations (11) and (12). By substituting $k_1 \equiv \frac{t_1 e^{\mu t_1}}{X_{t_1}} - 1$, $q_1 \approx (k_1 - 1)(1 - \sigma W(t_1))$ into equation (4), the stock price at any time $t \in [t_1, t_2]$ can be approximated by the lognormal diffusion process expressed as follows:

$$ S(t) \approx S(0)e^{\mu t - k_1 t + \sigma W(t_1) + \sigma (W(t) - W(t_1))}. $$

(13)

Thus, the stock return for the time interval $[t_1, t_2]$ can be expressed as a drifted Brownian motion and Theorem 2.1 can be applied to solve the joint density of the stock price at time

$$
\int_{\Delta} f_{M(T)}(u, \delta (T)) du
\Rightarrow
\frac{1}{\sqrt{2\pi \Delta}} e^{-\frac{(u - \Delta)^2}{2\Delta}}
$$

By applying Lemma 2.2, equation (8) can be rewritten as

$$ C = e^{-rT} \int_0^b \left( S(0)e^{\sigma W} - K \right) \times \left( \int_0^b 2(2m - w) e^{\sigma W - \frac{1}{2} \sigma^2 T + \frac{1}{\pi}(2m - w)^2} dm \right) dw
$$

$$ = e^{-rT} \int_0^b \left( - \frac{K}{\sqrt{2\pi T}} e^{-\frac{b^2 + aw - \frac{2a}{\sigma^2}}{2}} + \frac{S(0)}{\sqrt{2\pi T}} e^{-\frac{b^2 + aw + \frac{2a}{\sigma^2}}{2}} \right) dw. $$

(9)

In equation (9), each term of the integrand is of the form $L a_2 w^2 + a_1 w + a_0$ for some constants $a_0, a_1, a_2,$ and $L$. The following identity can convert the integrals of the aforementioned form into the cumulative distribution function (CDF) of the standard normal distribution by completing the square:

$$ \int_{-\infty}^{\infty} e^{a_2 x^2 + a_1 x + a_0} dx = \frac{\sqrt{\pi}}{a_2} e^{\frac{a_1^2 - 4a_0a_2}{4a_2^2}} N \left( \frac{1 - m}{s} \right). $$

(10)

where $a_2 < 0$ to ensure that the integral is finite, $m = -\frac{a_1}{2a_2}$, $s = \frac{1}{\sqrt{-2a_2}}$, and $N(\cdot)$ denotes the CDF of the standard normal distribution. Thus, the Reiner and Rubinstein (1991) pricing formula can be derived as a linear combination of tail probability values, which can be evaluated by the CDF of the standard normal distribution.
\[ t_2 \text{ and the maximum stock price for this time interval. The } \]
\[ \text{aforementioned procedure can be repeated to find the log-} \]
\[ \text{normal diffusion process that approximates the stock price} \]
\[ \text{process between the exdividend dates } t_2 \text{ and } t_3 \text{. Again, the} \]
\[ \text{discrete stock price jump due to the payout of the dividend } c_2 \]
\[ \text{is approximated by the accumulated price decrement caused} \]
\[ \text{by a continuous dividend yield } q_2 \text{ paid from time } t_1 \text{ to time } t_2; \]
\[ \text{that is,} \]
\[ S(t) = \left( S(t_1) e^{(t - t_1) + \sigma(W(t_2) - W(t_1)) - c_2} \right) \]
\[ \times e^{(t - t_2) + \sigma(W(t) - W(t_3))} \]
\[ = S(0) e^{(\mu - q_1)(t_1 + \sigma W(t_1))} \]
\[ \times e^{(t - t_2)(t_1 - t_1) + \sigma(W(t_2) - W(t_1))} e^{(t - t_2) + \sigma(W(t) - W(t_2))}. \]
\[ \text{(14)} \]
\[ \text{Note that } q_2 \text{ can be approximately solved by the first-order} \]
\[ \text{Taylor expansion to obtain} \]
\[ q_2 \approx \frac{(k_2 - 1) \{ 1 - k_1 \sigma W(t_1) - \sigma(W(t_2) - W(t_1)) \}}{t_2 - t_1}, \]
\[ \text{(15)} \]
\[ \text{where } k_2 \equiv \frac{e^{-\mu_2 + 1} - 1}{\sigma_2} - 1. \text{ Therefore, the stock price at} \]
\[ \text{time } t \in [t_2, t_3] \text{ can be approximated by a lognormal diffusion process} \]
\[ \text{by substituting equation (15) into equation (14) to obtain} \]
\[ S(t) \approx S(0) \]
\[ \times e^{(\mu - k_1 - k_2 + 2) + k_1 k_2 \sigma W(t_1) + k_2 \sigma(W(t_2) - W(t_1)) + \sigma(W(t) - W(t_2))}. \]
\[ \text{(16)} \]
\[ \text{Theorem 2.1 can again be used to derive the conditional joint density of random variables} \]
\[ \max_{t_1 \leq t \leq t_2} \left( \hat{W}_1(t - t_1) \right) \text{ and } \hat{W}_1(t_2 - t_1) \text{ based on the information of } \]
\[ S(t). \text{ The stock price process } \hat{S}(t) \text{ for the third time interval } t \in [t_2, t_3] \text{ can be reexpressed as} \]
\[ \hat{S}(t) = S''(0) e^{k_1 k_2 \sigma \hat{W}(t_1) + k_2 \sigma \hat{W}_1(t_1 - t_3) + \sigma \hat{W}_2(t - t_2)}, \]
\[ \text{(20)} \]
\[ \text{where } S''(0) \equiv S(0) e^{(\mu - k_1 - k_2 - 2) - k_1 k_2 \alpha - k_2 \mu(t_2 - t_1) - \alpha(t - t_2)}, \]
\[ \text{and } \hat{W}_2(t - t_2) \equiv \alpha(t - t_2) + (W(t) - W(t_2)). \text{ Note that both} \]
\[ \hat{W}_1(t_1) \text{ and } \hat{W}_1(t_1 - t_1) \text{ are } F_{t_2} \text{ measurable. Theorem 2.1} \]
\[ \text{can again be used to derive the conditional joint density of random variables} \]
\[ \max_{t_1 \leq t \leq t_2} \left( \hat{W}_2(t - t_2) \right) \text{ and } \hat{W}_2(t_2 - t_2) \text{ based on the information of } \]
\[ S(t). \text{ By combining equations (18), (19) and (20), the approximated stock price process } \]
\[ \hat{S}(t) \text{ can be rewritten as} \]
\[ \hat{S}(t) = \left\{ \begin{array}{ll}
S(0) e^{\sigma \hat{W}(t)} & 0 \leq t < t_1, \\
S(0) e^{k_1 \sigma \hat{W}(t_1) + \sigma \hat{W}_1(t_1 - t_1)} & t_1 \leq t < t_2, \\
S(0) e^{k_1 k_2 \sigma \hat{W}_1(t_1) + k_2 \sigma \hat{W}_1(t_2 - t_1) + \sigma \hat{W}_2(t_2 - t_2)} & t_2 \leq t < t_3.
\end{array} \right. \]
\[ \text{(21)} \]
\[ \text{3.2. Evaluate the integration of exponential functions} \]
\[ \text{The pricing formula in this paper can be expressed in terms of} \]
\[ \text{a multiple integration of an exponential function, the exponent} \]
\[ \text{term of which is a quadratic function of integrators. Theorem} \]
\[ \text{3.1 shows that such an integration can be expressed as a CDF} \]
\[ \text{of a multi-variant normal distribution by taking advantage of} \]
\[ \text{some matrix and vector calculations. For convenience, for any} \]
\[ \text{matrix } \Sigma, \text{ we use } |\Sigma|, \text{ and } \Sigma^{-1} \text{ to denote the determinant,} \]
\[ \text{the transpose, and the inverse of } \Sigma. \text{ If } i \text{ and } j \text{ are the element} \]
\[ \text{located in the } i-\text{th row and } j-\text{th column of } \Sigma. \text{ For any vector} \]
\[ \nu, \text{ we use } v_j \text{ to denote the } i-\text{th element of } \nu. \]
\[ \text{Theorem 3.1 Let } x \text{ and } B \text{ be an } n \times 1 \text{ constant vector, } C \text{ be a constant, and } \Lambda \text{ be an} \]
\[ \text{n } \times n \text{ symmetric invertible negative-definite constant matrix. For any general quadratic formula} \]
\[ x^T A x + B^T x + C, \text{ the } n \text{-variate integral for } e^{x^T A x + B^T x + C} \]
\[ \text{can be expressed in terms of a CDF of an } n \text{-dimensional standard normal distribution } F_{Y_1, Y_2, ..., Y_n} \text{ with covariance} \]
\[ \text{matrix } \Sigma \text{ as follows:} \]
\[ e^{c'} \rho \sqrt{-i} \left| A \right| \]
\[ \times \left( \begin{array}{c}
\rho_1 - m_1 \\
\rho_2 - m_2 \\
\vdots \\
\rho_n - m_n
\end{array} \right)
\]
\[ \text{where the vector } m = (m_1, m_2, ..., m_n) \equiv -\frac{1}{2} A^{-1} B, \text{ and } \Sigma = -2 \Lambda^{-1} \text{, and } S \text{ is a } n \times n \text{ diagonal matrix defined as} \]
\[ S_{i,j} = \begin{cases}
\rho_1 \rho_2 \rho_n & \text{if } i = j, \\
0 & \text{otherwise.}
\end{cases} \]
\[ \text{Proof See Appendix A.} \]
\[ \square \]
\[ 4. \text{ Analytical formulae} \]
\[ \text{We will first derive the approximating analytical pricing} \]
\[ \text{formula for the up-and-out barrier call with a single discrete} \]
dividend in section 4.1. This approach can be extended to derive the pricing formula for the multi-dividend case as discussed in section 4.2.

4.1. The single-discrete-dividend case

Since only one dividend is paid during the life of the option, the option maturity date \( T \) is later than the first exdividend date \( t_1 \) but earlier than the second exdividend date \( t_2 \) (i.e. \( t_1 < T < t_2 \)). The call option value \( C \) can be evaluated by applying the risk-neutral valuation method to the approximated stock price process \( \hat{S}(t) \) defined in equation (21) as follows:

\[
\hat{C} = e^{-rT}E\left[(\hat{S}(T) - K)1_{\{\hat{E}_1 \cap \hat{E}_2 \cap \hat{E}_3\}}\right], \tag{24}
\]

where \( \hat{E}_1, \hat{E}_2 \) denote the events that the stock price process does not hit the barrier \( B \) during the time intervals \([0, t_1)\) and \([t_1, T)\), respectively, and \( \hat{E}_3 \) denotes the event that the option is in the money at the maturity date. Specifically, the three events \( \hat{E}_1, \hat{E}_2 \) and \( \hat{E}_3 \) are defined as follows:

\[
\begin{align*}
\hat{E}_1 & = \{ \hat{S}(t) < B, \quad \forall t \in [0, t_1) \}, \\
\hat{E}_2 & = \{ \hat{S}(t) < B, \quad \forall t \in [t_1, T) \}, \\
\hat{E}_3 & = \{ \hat{S}(T) > K \}. \tag{25}
\end{align*}
\]

To evaluate the option value, we derive the joint density of the maximum stock prices over the time interval \([0, t_1)\) and \([t_1, T)\) and the stock price at time \( t_1 \) and \( T \) by Theorem 2.1. Define \( \hat{M}(t_1) = \max_{s \leq t_1} \hat{W}(t) \) as the maximum value of \( \hat{W}(t) \) over the time interval \([0, t_1)\), and \( \hat{M}_1(T - t_1) = \max_{t_1 \leq s \leq T} \hat{W}_1(t_1 - s) \) as the maximum value of \( \hat{W}_1(t_1 - s) \) over the time interval \([t_1, T)\). Thus, the three events \( \hat{E}_1, \hat{E}_2 \) and \( \hat{E}_3 \) can be rewritten by substituting the definition of \( \hat{S}(t) \) defined in equation (21) into equation (25) to obtain

\[
\begin{align*}
\hat{E}_1 & = \{ \hat{S}(0)e^{\sigma \hat{M}(t_1)} < B \} = \{ \hat{M}(t_1) < b \}, \\
\hat{E}_2 & = \{ \hat{S}(0)e^{\sigma \hat{W}(t_1) + \sigma \hat{M}_1(T - t_1)} < B \} = \{ \hat{M}_1(T - t_1) < b' - k_1 \hat{W}(t_1) \}, \\
\hat{E}_3 & = \{ \hat{S}(0)e^{\sigma \hat{W}(t_1) + \sigma \hat{W}_1(T - t_1)} > K \} = \{ \hat{W}_1(T - t_1) > k' - k_1 \hat{W}(t_1) \},
\end{align*}
\]

where \( b, b' \) and \( k \) represent \( \frac{1}{\sigma} \log \frac{B}{M(0)} \), \( \frac{1}{\sigma} \log \frac{B}{\hat{S}(0)} \) and \( \frac{1}{\sigma} \log \frac{K}{\hat{S}(0)} \), respectively. The joint density functions \( \hat{f}_{\hat{M}(t_1), \hat{W}(t_1)} \) and \( \hat{f}_{\hat{M}_1(T - t_1), \hat{W}_1(T - t_1)} \) can be derived from Theorem 2.1 as follows:

\[
\begin{align*}
\hat{f}_{\hat{M}(t_1), \hat{W}(t_1)}(m, w) & = \frac{2(2m - w)\sigma w - \frac{1}{2}a^2t_1 - \frac{1}{2}(2m - w)^2}{t_1\sqrt{2\pi t_1}} 1_{\{m \geq w^+, \quad \text{otherwise}\}}, \tag{26} \\
\hat{f}_{\hat{M}_1(T - t_1), \hat{W}_1(T - t_1)}(m_1, m) & = \frac{2(2m_1 - w_1)\sigma w_1 - \frac{1}{2}a^2(T - t_1) - \frac{1}{2}(2m_1 - w_1)^2}{(T - t_1)\sqrt{2\pi(T - t_1)}} \times e^{a w_1 - \frac{1}{2}a^2(T - t_1)} \times 1_{\{m_1 \geq w_1^+, \quad \text{otherwise}\}}. \tag{27}
\end{align*}
\]

For simplicity, we will use the symbols \( \hat{f}_0 \) and \( \hat{f}_1 \) to represent \( \hat{f}_{\hat{M}(t_1), \hat{W}(t_1)} \) and \( \hat{f}_{\hat{M}_1(T - t_1), \hat{W}_1(T - t_1)} \), respectively. Since the two drifted Brownian motions \( \hat{W}(t) \) for \( t \in [0, t_1) \) and \( \hat{W}_1(t_1 - s) \) for \( t \in [t_1, T) \) are independent due to the Markov property of the Brownian motion, the joint density function of maximum stock prices over \([0, t_1)\) and \([t_1, T)\) and the stock prices at time \( t_1 \) and \( T \) can be calculated by directly multiplying \( \hat{f}_0 \) by \( \hat{f}_1 \). By substituting this joint density function into equation (24), the analytical pricing formula can be derived by the law of iterated expectation as follows:

\[
\hat{C} = e^{-rT}E\left[\left(\hat{S}(T) - K\right)1_{\{\hat{E}_1 \cap \hat{E}_2 \cap \hat{E}_3\}}\hat{W}(t_1), \hat{M}(t_1)\right]\]

\[
e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{b} \int_{-\infty}^{\infty} \int_{-\infty}^{b' - k_1 w} \int_{-\infty}^{k' - k_1 w} \left(\hat{S}(0)e^{k_1\sigma w + \sigma w_1} - K\right) \hat{f}_1(m_1, m_1 \hat{f}_0(m, w) dm_1 dw_1 dm w \tag{28} \right.

\[
e^{-rT} \int_{-\infty}^{b} \int_{-\infty}^{b} \int_{-\infty}^{b' - k_1 w} \int_{-\infty}^{k_1 w} \left(\hat{S}(0)e^{k_1\sigma w + \sigma w_1} - K\right) \hat{f}_1(m_1, m) \hat{f}_0(m, w) dm_1 dm_1 dm w \tag{29}
\]

where the domain of the integral in equation (29) is obtained by mimicking the analysis in figure 1; it is derived by taking the intersection of the supports of \( \hat{f}_1(m_1, m_1) \) and \( \hat{f}_0(m, w) \) with the integral domain in equation (28). In the integrand in equation (29), only \( \hat{f}_0(m, w) \) contains the integrator \( m \) and \( \hat{f}_1(m_1, w_1) \) contains the integrator \( m_1 \). Therefore, \( \int_{-\infty}^{b} \hat{f}_0(m, w) dm \) and \( \int_{-\infty}^{b' - k_1 w} \hat{f}_1(m_1, w_1) dm_1 \) can be integrated separately by Lemma 2.2 as follows:

\[
\hat{C} = e^{-rT} \int_{-\infty}^{b} \int_{-\infty}^{b' - k_1 w} \left(\hat{S}(0)e^{k_1\sigma w + \sigma w_1} - K\right) \hat{f}_1(m_1, m_1 \hat{f}_0(m, w) dm_1 dw_1 dm w \tag{30}
\]

The variables in the lower and the upper limits for the above integral can be eliminated by the change of variables; that is, we substitute \( x = w_1 + k_1 w \) and \( y = w \) into equation (30) to obtain†

\[
\hat{C} = e^{-rT} \int_{-\infty}^{b} \int_{-\infty}^{b' - k_1 w} \left(\hat{S}(0)e^{k_1\sigma x + \sigma y} - K\right) \hat{f}_1(m_1, m_1 \hat{f}_0(m, w) dm_1 dw_1 dm w \tag{31} \right.

\[
\times \frac{1}{\sqrt{2\pi(T - t_1)}} e^{\frac{1}{2}a^2(x - k_1 y) - \frac{1}{2}(2m_1 - w_1)^2} \int_{-\frac{1}{a^2}(x - k_1 y)}^{\frac{1}{a^2}(x - k_1 y)} e^{\frac{a^2}{2}(T - t_1)} \times \left(1 - e^{-\frac{a^2(x - k_1 y)}{T - t_1}}\right) \right)
\]

†Note that the Jacobian determinant \( \frac{\partial (w_1, w)}{\partial (x, y)} = 1 \).
4.1 Corollary

\[ I(i) = - \frac{K}{2\pi\sqrt{\Delta}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( \frac{2\alpha(x-y)^2}{\Delta} \right) \phi \left( \frac{x}{\sqrt{2\alpha}} \right) \phi \left( \frac{y}{\sqrt{2\alpha}} \right) \, dx \, dy, \]

where the integrand \( I(1) \) is defined as

\[ I(1) = - \frac{K}{2\pi\sqrt{\Delta}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( \frac{2\alpha(x-y)^2}{\Delta} \right) \phi \left( \frac{x}{\sqrt{2\alpha}} \right) \phi \left( \frac{y}{\sqrt{2\alpha}} \right) \, dx \, dy. \]

\[ I(2) = -I(1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi \left( \frac{x}{\sqrt{2\alpha}} \right) \phi \left( \frac{y}{\sqrt{2\alpha}} \right) \, dx \, dy, \]

\[ I(3) = -I(1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi \left( \frac{x}{\sqrt{2\alpha}} \right) \phi \left( \frac{y}{\sqrt{2\alpha}} \right) \, dx \, dy, \]

\[ I(4) = I(1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi \left( \frac{x}{\sqrt{2\alpha}} \right) \phi \left( \frac{y}{\sqrt{2\alpha}} \right) \, dx \, dy. \]

Since each exponent term of the integrands \( I(1), I(2), \ldots, I(8) \) is a quadratic polynomial of integrators \( x \) and \( y \), the double integral of each integrand can be expressed in terms of the CDF of a bivariate normal distribution by the following Corollary:

**Corollary 4.1** The double integral \( G \) with the following format can be expressed in terms of the CDF of a bivariate standard normal distribution \( F_{Y_1,Y_2} \) as follows:

\[ G(p, q, a_1, a_2, a_3, a_4, a_5, a_6) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ax^2 + bxy + cy^2 + dx + ey} \, dx \, dy \]

\[ = 2\pi \exp \left( a_6 + \frac{a_2a_5a_3 - a_3a_4^2 - a_1a_5^2}{\Delta} \right) \times F_{Y_1,Y_2} \left( \frac{\Delta p + 2a_1q - 2a_3}{\sqrt{-2a_3}}, \frac{\Delta q + 2a_1p - 2a_3}{\sqrt{-2a_3}} ; \Sigma \right), \]

where \( \Delta = 4a_1a_3 - a_2^2 \), and \( \Sigma \equiv \left( \begin{array}{cc} 1 & \frac{a_2}{2\sqrt{a_1a_3}} \\ \frac{a_2}{2\sqrt{a_1a_3}} & \frac{2a_1e + 2a_2}{4a_3} \end{array} \right) \).

Proof It can be easily derived from Theorem 3.1 and is proved in Appendix B.

With Corollary 4.1, the double integrals of \( I(1), I(2), \ldots, I(8) \) can be converted into CDFs of bivariate normal distributions. For example, the integral of \( I(1) \) can be rewritten as

\[ \int_{-\infty}^{b} \int_{k}^{b} I(1) \, dx \, dy = \int_{-\infty}^{b} \int_{k}^{b} - \frac{K}{2\pi\sqrt{\Delta}} \exp \left( \frac{2\alpha(x-y)^2}{\Delta} \right) \, dx \, dy \]

\[ = - \frac{K}{2\pi\sqrt{\Delta}} \left( \int_{-\infty}^{b} \int_{k}^{b} \exp \left( \frac{2\alpha(x-y)^2}{\Delta} \right) \, dx \, dy \right). \]

Define \( a_1(1) = - \frac{1}{2\pi\sqrt{\Delta}} \), \( a_2(1) = \frac{k}{\pi\alpha} \), \( a_3(1) = - \frac{k^2}{\pi\alpha} + \frac{1}{\pi\alpha} \), \( a_4(1) = \alpha \), \( a_5(1) = \alpha - \frac{1}{\pi\alpha} \), \( a_6(1) = \frac{1}{\pi\alpha} \), \( a_7(1) = \frac{1}{\pi\alpha} \), \( a_8(1) = \frac{1}{\pi\alpha} \), \( a_9(1) = \frac{1}{\pi\alpha} \), and \( a_{10}(1) = \frac{1}{\pi\alpha} \). These four terms are obtained by multiplying the strike price \( K \) (in the first line) by the terms in the following two lines of equation (31). I(5), I(6), I(7) and I(8), which are obtained by multiplying \( S(0)e^{\sigma x} \) in the first line by the terms in the following two lines of equation (31), are defined as

\[ I(i) = - \frac{S(0)}{K} I(i - 4)e^{\sigma x}, \quad i = 5, \ldots, 8. \]

Consequently, we rewrite equation (36) as follows:

\[ D(1)G(b', b, a_1(1), a_2(1), a_3(1), a_4(1), a_5(1), a_6(1)) \]

\[ = G(k', b, a_1(1), a_2(1), a_3(1), a_4(1), a_5(1), a_6(1)), \]

where \( D(1) = - \frac{K}{2\pi\sqrt{\Delta}} \), and \( G \) is defined in equation (34). Similarly, the double integrals for \( I(2), I(3), \ldots, I(8) \) in equation (32) can all be expressed as

\[ \frac{1}{2\sqrt{a_1a_3}} \frac{a_2}{a_1a_3} \]
\[
\int_{-\infty}^{b} \int_{k'}^{b'} I(i) dxdy = D(i)G(b', b, a_1(i), a_2(i), a_3(i), a_4(i), a_5(i), a_6(i)) - G(k', b, a_1(i), a_2(i), a_3(i), a_4(i), a_5(i), a_6(i)),
\]
where \(a_1(i), a_2(i), a_3(i), a_4(i), a_5(i), a_6(i)\) and \(a_6(i)\) denote the coefficients of \(x^2, y^2, y, x, y\) and the constant term of the exponential term of \(I(i)\). Specifically, the parameters are given by \(a_1(i) = a_1(1), a_2(i) = a_2(1), a_2(i) = \left(1 - 1\right)^i a_2(1)\) for \(i = 2, 3, \ldots, 8\), and \(D(i), a_4(i), a_5(i)\) and \(a_6(i)\) are given by the following table:

<table>
<thead>
<tr>
<th>(i)</th>
<th>(D(i))</th>
<th>(a_4(i))</th>
<th>(a_5(i))</th>
<th>(a_6(i))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(\frac{K}{2\pi \sqrt{(T-t_1)T}})</td>
<td>(\frac{2b'}{T-t_1} + \alpha)</td>
<td>(\alpha + k_1\left(\frac{2b'}{T-t_1} - \alpha\right))</td>
<td>(-\frac{4b'^2}{T^2} + \frac{2\alpha^2}{T} - \frac{T^2 \alpha^2}{2})</td>
</tr>
<tr>
<td>3</td>
<td>(\frac{K}{2\pi \sqrt{(T-t_1)T}})</td>
<td>(\alpha)</td>
<td>(\frac{2b}{t_1} + \frac{1}{\alpha - ak_1})</td>
<td>(\frac{2b^2}{T-t_1} - \frac{T^2 \alpha^2}{2})</td>
</tr>
<tr>
<td>4</td>
<td>(-\frac{K}{2\pi \sqrt{(T-t_1)T}})</td>
<td>(\frac{2b'}{T-t_1} + \alpha)</td>
<td>(\frac{2b}{t_1} + \frac{1}{\alpha - ak_1})</td>
<td>(\frac{2b^2}{T-t_1} - \frac{T^2 \alpha^2}{2})</td>
</tr>
<tr>
<td>5</td>
<td>(-\frac{2n}{2\pi \sqrt{(T-t_1)T}})</td>
<td>(\alpha + \sigma)</td>
<td>(\sigma)</td>
<td>(-\frac{4b'^2}{T^2} + \frac{2\alpha^2}{T} - \frac{T^2 \alpha^2}{2})</td>
</tr>
<tr>
<td>6</td>
<td>(-\frac{2n}{2\pi \sqrt{(T-t_1)T}})</td>
<td>(\frac{2b'}{T-t_1} + \alpha + \sigma)</td>
<td>(\alpha + k_1\left(\frac{2b'}{T-t_1} - \alpha\right))</td>
<td>(\frac{2b^2}{T-t_1} - \frac{T^2 \alpha^2}{2})</td>
</tr>
<tr>
<td>7</td>
<td>(-\frac{2n}{2\pi \sqrt{(T-t_1)T}})</td>
<td>(\alpha + \sigma)</td>
<td>(\frac{2b}{t_1} + \frac{1}{\alpha - ak_1})</td>
<td>(\frac{2b^2}{T-t_1} - \frac{T^2 \alpha^2}{2})</td>
</tr>
<tr>
<td>8</td>
<td>(-\frac{2n}{2\pi \sqrt{(T-t_1)T}})</td>
<td>(\alpha + \sigma)</td>
<td>(\frac{2b}{t_1} + \frac{1}{\alpha - ak_1})</td>
<td>(\frac{2b^2}{T-t_1} - \frac{T^2 \alpha^2}{2})</td>
</tr>
</tbody>
</table>

Thus, the option pricing formula in equation (32) can be rewritten as

\[
\tilde{C} = e^{-rT} \int_{-\infty}^{b} \int_{k'}^{b'} I(1) + I(2) + \cdots + I(8) dxdy = e^{-rT} \sum_{i=1}^{8} D(i)G(b', b, a_1(i), a_2(i), a_3(i), a_4(i), a_5(i), a_6(i)) - G(k', b, a_1(i), a_2(i), a_3(i), a_4(i), a_5(i), a_6(i)).
\]

Note that if the upper barrier \(B\) tends to infinity, the up-and-out call degenerates into a vanilla call option. Indeed, both \(b = \left(1 - \frac{1}{\alpha} \log \frac{B}{S_0}\right)\) and \(b' = \left(1 - \frac{1}{\alpha} \log \frac{B}{S_0}\right)\) tend to infinity as \(B \to \infty\), and our pricing formula degenerates into the approximating formula for pricing vanilla stock call options with one discrete dividend derived in Dai and Lyuu (2009).

### 4.2. Multi-discrete-dividend case

The above approach can be repeatedly applied to derive approximated pricing formulae for barrier stock options with multiple discrete dividends. For simplicity, we derive the pricing formula for the two-dividend case in this section. The extensions for cases involving three or more dividends are straightforward. Note that \(t_1 < t_2 < T < t_3\) in the two-dividend case.

To evaluate the option, we need to derive the joint density function of the maximum stock prices over the time intervals \([0, t_1), [t_1, t_2)\) and \([t_2, T]\) and the stock price at the maturity date \(T\). Define \(\tilde{M}_1(t_2-t_1) \equiv \max_{t \leq t_2} \tilde{W}_1(t-t_1)\) as the maximum value of \(\tilde{W}_1(t)\) over the time interval \([t_1, t_2)\), and \(\tilde{M}_2(T-t_2) \equiv \max_{t \geq t_2} \tilde{W}_2(t-t_2)\) as the maximum value of \(\tilde{W}_2(t)\) over the time interval \([t_2, T]\). The joint density function of \(\tilde{M}_1(t_2-t_1)\) and \(\tilde{W}_1(t_2-t_1)\), and the joint density function of \(\tilde{M}_2(T-t_2)\) and \(\tilde{W}_2(T-t_2)\) can be derived by applying Theorem 2.1 as follows:

\[
\begin{align*}
\tilde{f}_{\tilde{M}_1(t_2-t_1), \tilde{W}_1(t_2-t_1)}(m_1, w_1) &= \frac{e^{-r(T-t_1)2\pi \sqrt{(T-t_1)T}}}{\sqrt{2\pi T-t_1}T-t_1} \times e^{\frac{1}{4T^2}(T-t_1)^2 - \frac{1}{2T}T-t_1^2} \left(\frac{m_1-w_1}{2T}\right)^2, \\
& \quad \text{if } m_1 \geq w_1, \quad \text{otherwise, (37)}
\end{align*}
\]

\[
\begin{align*}
\tilde{f}_{\tilde{M}_2(T-t_2), \tilde{W}_2(T-t_2)}(m_2, w_2) &= \frac{e^{-r(T-t_2)2\pi \sqrt{(T-t_2)T}}}{\sqrt{2\pi T-t_2}T-t_2} \times e^{\frac{1}{4T^2}(T-t_2)^2 - \frac{1}{2T}T-t_2^2} \left(\frac{m_2-w_2}{2T}\right)^2, \\
& \quad \text{if } m_2 \geq w_2, \quad \text{otherwise.}
\end{align*}
\]

For simplicity, we use \(\tilde{f}_0, \tilde{f}_1\) and \(\tilde{f}_2\) to represent the density functions \(\tilde{f}_{\tilde{M}_1(t_1), \tilde{W}_1(t_1)}\) (see equation (26)), \(\tilde{f}_{\tilde{M}_2(t_2-t_1), \tilde{W}_1(t_1-t_1)}\) and \(\tilde{f}_{\tilde{M}_2(T-t_2), \tilde{W}_2(T-t_2)}\), respectively. Note that the drifted Brownian motions \(\tilde{W}(t)\) for \(t \in [0, t_1)\), \(\tilde{W}_1(t_1-t_1)\) for \(t \in [t_1, t_2)\) and \(\tilde{W}_2(t-t_2)\) for \(t \in [t_2, t_3]\) are independent due to the Markov property of the Brownian motion; therefore, the joint density function of maximum stock prices over \([0, t_1), [t_1, t_2]\) and \([t_2, T]\) and the stock prices at time \(t_1, t_2\) and \(T\) can be calculated by directly multiplying \(\tilde{f}_0\) by \(\tilde{f}_1\) and \(\tilde{f}_2\).

The option value can be evaluated by the risk-neutral variation method as follows:

\[
\tilde{C} = e^{-rT} \mathbb{E} \left[ \left(\tilde{S}(T) - K\right) \mathbb{1}_{\tilde{E}_1 \cap \tilde{E}_2 \cap \tilde{E}_3 \cap \tilde{E}_4} \right],
\]

where \(\tilde{E}_1, \tilde{E}_2\) and \(\tilde{E}_3\) represent the events that the stock price process does not hit the barrier \(B\) during the time intervals \([0, t_1), [t_1, t_2)\) and \([t_2, T]\), respectively, and \(\tilde{E}_4\) denotes the event that the stock price at maturity is greater than the strike price. Specifically, \(\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\) and \(\tilde{E}_4\) are defined as
where $k^* \equiv \frac{1}{2} \log \frac{K}{S}$, and $b^* \equiv \frac{1}{2} \log \frac{B}{S}$, respectively.

Thus, we can compute the pricing formula in equation (38) by applying the law of iterated expectation as follows:

\[
\tilde{C} = e^{-rT} E \left[ E \left[ \left( \hat{S}(T) - K \right) 1_{ \{ E_1 \cap E_2 \cap E_3 \cap E_4 \} } \right] \left( \tilde{W}(t_1), \hat{M}(t_1), \tilde{W}(t_2) - \hat{M}(t_2 - t_1) \right) \tilde{W}(t_1), \hat{M}(t_1) \right] \right]
\]

\[
\tilde{C} = e^{-rT} \int_{-\infty}^{b} \int_{-\infty}^{b} \int_{-\infty}^{b} \int_{-\infty}^{b} \int_{-\infty}^{b} \left( S(0) e^{\sigma \tilde{w}_1 + \tilde{w}_2 - \hat{w}} - K \right) \tilde{f}_2(m_2, w_2) \tilde{f}_1(m_1, w_1) \tilde{f}_0(m, w) dm_2 dw_2 dm_1 dw_1 dm w
\]

(40)

where the domain of the integral in equation (41) is obtained by taking the intersection of the supports of $\tilde{f}_2(m_2, w_2)$, $\tilde{f}_1(m_1, w_1)$ and $\tilde{f}_0(m, w)$ with the integral domain in equation (40). Since only $\tilde{f}_2(m_2, w_2)$ contains the integrator $m_2$, $\tilde{f}_1(m_1, w_1)$ contains $m_1$, and $\tilde{f}_0(m, w)$ contains $m$ in the integrand in equation (41), $\tilde{f}_0(m, w)dm$, $\tilde{f}_1(m_1, w_1)dm_1$ and $\tilde{f}_2(m_2, w_2)dm_2$ can be simplified by applying Lemma 2.2 as follows:

\[
\tilde{C} = e^{-rT} \int_{-\infty}^{b} \int_{-\infty}^{b} \int_{-\infty}^{b} \int_{-\infty}^{b} \int_{-\infty}^{b} \left( S(0) e^{\frac{1}{2} \sigma \tilde{w}_1 + \tilde{w}_2 - \hat{w}} - K \right) \tilde{f}_2(m_2, w_2) \tilde{f}_1(m_1, w_1) \tilde{f}_0(m, w) dm_2 dw_2 dm_1 dw_1 dm w
\]

(41)

To eliminate the variables in the lower and the upper limits for the integrals on $w_1$ and $w_2$, the equations $x = w_1 + k_1 w_1 + k_1 k_2 w_2$, $y = w_1 + k_1 w$ and $z = w$ are substituted into the aforementioned formula to obtain

\[
\tilde{C} = e^{-rT} \int_{-\infty}^{b} \int_{-\infty}^{b} \int_{-\infty}^{b} \left( S(0) e^{\sigma x - K} \right) \times
\]

\[
\frac{1}{\sqrt{2\pi(T-t_1)}} e^{\alpha(x-k_2 y)-\frac{1}{2} \alpha^2(T-t_2) - \frac{(y-k_2 z)^2}{2\alpha^2(T-t_2)}}
\]

\[
\times \int_{-\infty}^{b} \left( 1 - e^{-\frac{1}{2\pi}(T-t_2)} \right) \left( 2 \alpha^2 (T-t_2) \right) \left( 1 - e^{-\frac{2\alpha^2(T-t_2)}{2\pi}} \right) dx dy dz
\]

(42)

where $J(1), J(2), \ldots, J(8)$ are defined in table 1, and $J(9), J(10), \ldots, J(16)$ are defined as

\[
J(i) = - \frac{S(0)}{K} J(i - 8) e^{\sigma i}, \quad i = 9, \ldots, 16.
\]

Since the exponent term of each of the integrands $J(1), J(2), \ldots, J(16)$ is a quadratic form of the integrators $x, y$ and $z$, the triple integral of each integrand can be expressed in terms of a trivariate normal CDF by the following corollary:

**Corollary 4.2** The triple integral with the following format can be expressed in terms of a CDF of a trivariate standard normal distribution $F_{Y_1, Y_2, Y_3}$ as follows:

\[
pq \cdot r \cdot d \cdot a_1, d_2, \ldots, \cdot \cdot \cdot, a_{10} \Rightarrow \int_{-\infty}^{b} \int_{-\infty}^{b} \int_{-\infty}^{b} e^{\alpha x^2 + \alpha y^2 + \alpha z^2 + \alpha^2 x y + \alpha^2 y z + \alpha^2 x z + \alpha^2 x y z} dxdydz
\]

\[
= C \left[ \left( \begin{array}{c} \pi^3 \end{array} \right) \left( \begin{array}{c} p_1 - m_{11} \frac{S_{1,1}}{S_1} \frac{p_2 - m_{22} \frac{S_{2,2}}{S_2}}{S_2} \frac{p_3 - m_{33} \frac{S_{3,3}}{S_3}}{S_3} \end{array} \right) \right),
\]

(43)

\[
A = \left( \begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_6 & a_7 & a_8 \end{array} \right), \quad \frac{S_{1,1}}{S_1} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & S_{2,2} \\ 0 & 0 & S_{3,3} \end{array} \right), \quad \Sigma = (-2 \Sigma A)^{-1}.
\]

\[
S_{ij,j} = \sqrt{\left( \sum_{i,j,k} (-2A)^{-1} \right)} m = -\frac{1}{2} A^{-1} B, \quad C = a_{10} - \frac{1}{2} B^T A^{-1} B, \quad \text{and} \quad \Sigma = (-2 \Sigma A)^{-1}.
\]

\*Note that the Jacobian determinant $\frac{\partial(w_1, w_2, w)}{\partial(x, y, z)} = 1.$

\*Note that the Jacobian determinant $\frac{\partial(w_1, w_2, w)}{\partial(x, y, z)} = 1.$
Define \( \zeta \) as \( \sqrt{8\pi^3(T-t_2)(t_2-t_1)t_1} \), and \( \eta \) as \( -\frac{z^2}{2t_1} + \alpha z + \alpha(y-zk_1) + \alpha(x-yk_2) - \frac{\alpha^2 t_1}{2} - \frac{1}{2}\alpha^2 (T-t_2) - \frac{1}{2}\alpha^2 (t_2-t_1) - \frac{(x-yk_2)^2}{2(T-t_2)} - \frac{(y-zk_1)^2}{2(t_2-t_1)} \).

\[
J(1) = -\frac{K}{\zeta} e^{\eta}, \quad J(2) = \frac{K}{\zeta} e^{\eta + \frac{2 b (T-b)}{t_1}}, \\
J(3) = \frac{K}{\zeta} e^{\eta + \frac{2 b (T-b)}{t_1^2}} \int_{t_2}^{b} e^{-\frac{b}{t_1}} (b' - z^2) \ dx, \\
J(4) = -\frac{K}{\zeta} e^{\eta + \frac{2 b (T-b)}{t_1}} \int_{t_2}^{b} e^{-\frac{b}{t_1}} (b' - z^2) \ dx, \\
J(5) = \frac{K}{\zeta} e^{\eta + \frac{2 b (T-b)}{t_1^2}} \int_{t_2}^{b} e^{-\frac{b}{t_1}} (b' - z^2) \ dx, \\
J(6) = -\frac{K}{\zeta} e^{\eta + \frac{2 b (T-b)}{t_1}} \int_{t_2}^{b} e^{-\frac{b}{t_1}} (b' - z^2) \ dx, \\
J(7) = \frac{K}{\zeta} e^{\eta + \frac{2 b (T-b)}{t_1^2}} \int_{t_2}^{b} e^{-\frac{b}{t_1}} (b' - z^2) \ dx, \\
J(8) = \frac{K}{\zeta} e^{\eta + \frac{2 b (T-b)}{t_1}} \int_{t_2}^{b} e^{-\frac{b}{t_1}} (b' - z^2) \ dx.
\]

5. Numerical results

Unlike most numerical pricing approaches that might generate unstable pricing results or hedging parameters (i.e. the Greek letters) as mentioned in Figlewski and Gao (1999) and Dai and Lyuu (2010), our approximate pricing formulae can generate stable pricing results and hedging parameters as illustrated in figure 2. In panel (a), an up-and-out call option value increases with the increment of the initial stock price when the stock price is low. However, the increment of the initial stock price also increases the probability for the option to knock out (i.e. the stock price path goes upward to reach the barrier). Therefore, when the initial stock price is higher than a certain level, say 52 in this numerical example, the option value decreases with the increment of the initial stock price. This phenomenon can be confirmed by checking the delta, i.e. the rate of change of the option price with the price of the underlying stock, as illustrated in panel (b). The delta smoothly decreases with the increment of the initial stock price and becomes negative when the stock price exceeds 52.

To examine the superiority of our pricing formulae, we will compare the accuracy among our approximation pricing formula and other approximation formulae in the following tables. Ours denote the values generated by the approximation pricing formulae proposed in this paper. We also follow the Chiras and Manaster (1978) assumption by approximating the discrete dividends paid over the life of the option with the equivalent continuous dividend yield \( q \) being derived as follows:

\[
S(0)e^{-qt} = S(0) - \sum_{i=1}^{n} c_i e^{-rt_i},
\]

where \( n \) denotes the number of dividends paid during the life of the option. Then the discrete-dividend barrier option can be approximately priced by the barrier option pricing formula with a continuous dividend yield proposed by Reiner and Rubinstein (1991), and the pricing results generated by this approach are listed under the ContDiv columns. Besides, we can follow Model 1 (see Roll 1977) by assuming that the process of the net-of-dividend stock price \( S_N(t) \) follows a lognormal diffusion process. In addition, the initial net-of-dividend stock price is defined as

\[
S_N(0) = S(0) - \sum_{i=1}^{n} c_i e^{-rt_i}.
\]

Table 1. The definitions of \( J(1), J(2), \ldots, J(8) \).

\[
\begin{array}{ll}
\text{Function} & \text{Letters} \text{ as mentioned in Figlewski and Gao (1999) and Dai and Lyuu (2010), our approximate pricing formulae can generate stable pricing results and hedging parameters as illustrated in figure 2. In panel (a), an up-and-out call option value increases with the increment of the initial stock price when the stock price is low. However, the increment of the initial stock price also increases the probability for the option to knock out (i.e. the stock price path goes upward to reach the barrier). Therefore, when the initial stock price is higher than a certain level, say 52 in this numerical example, the option value decreases with the increment of the initial stock price. This phenomenon can be confirmed by checking the delta, i.e. the rate of change of the option price with the price of the underlying stock, as illustrated in panel (b). The delta smoothly decreases with the increment of the initial stock price and becomes negative when the stock price exceeds 52.}
\end{array}
\]

†Frishling (2002) argues that Model 1 could incorrectly render a down-and-out barrier option worthless because the net-of-dividend stock price may reach the barrier for a large present value of future dividend payments.

\[
\begin{array}{ll}
\text{Downloaded by \[National Chiao Tung University \] at 18:13 24 December 2014}
\end{array}
\]
Table 2 illustrates how the changes in the initial stock prices influence the option values and the accuracy of the aforementioned three pricing formulae. Similar to figure 2(a), the option value first increases and then decreases with the increment of the initial stock price. Similar to the phenomenon observed in Frishling (2002) for pricing vanilla options, different dividend-approximation models would generate very different prices. Here, we use a Monte Carlo simulation (denoted as MC) with 1,000,000 trials and a binomial lattice† (denoted as $L$) as proxy benchmarks. Recall that Frishling (2002), Bender and Vorst (2001), and Bos and Vandermark (2002) argue that only Model 3 can reflect the reality and generate consistent option prices. Thus, we use the Monte Carlo simulation as the first benchmark since it can faithfully model the downward jumps of the underlying stock price defined in Model 3. However, Baldi et al. (1999) argue that it might be difficult to obtain very precise results with the Monte Carlo simulation. Thus, we add the binomial lattice as another benchmark. It can be observed that the benchmark values produced by these two methods are close and coherent. By using the Monte Carlo simulation as the benchmark, it can be observed that our formula is more accurate than the other two formulae since the maximum absolute error (MAE) 0.0089 and the root-mean-squared error (RMSE) 0.0054 are lower than those for the other two formulae. In addition, the pricing errors of Model 1 are much smaller than the errors of the other two formulae. Model 2 produces very inaccurate results (the percentage of error $= 0.1765\approx 90\%$) when the initial stock price is high, say, 64. Using the binomial lattice as the benchmark also produces the same result. The MAE and the RMSE for ContDiv are 0.0169 and 0.01076, respectively‡. These two values for Model 1 are 0.1832 and 0.1163, respectively. They are much higher than the MAE 0.0013 and the RMSE 0.0010 for Ours. For simplicity, we will only use the Monte Carlo simulation with 1,000,000 trials as the benchmark (denoted by Benchmark) in the following experiments.

Table 3 compares the pricing results under different amounts of discrete dividend payout. It can be observed that the pricing errors of both ContDiv and Model 1 increase with the amount of the dividend payout, while the pricing errors of Ours are much smaller. MAE and RMSE of Ours are also smaller than those of ContDiv and Model 1. Table 4 illustrates the pricing results under different stock price volatilities. Note that the value of an up-and-out call decreases with the increment in the stock price volatility since a higher volatility implies a higher ‘knock out’ probability. It can also be observed that MAE and RMSE of Ours are all smaller than $10^{-2}$, while MAE and RMSE of both ContDiv and Model 1 are much higher. Table 5 analyses the impacts of changing the exdividend date on the option value. By observing the Benchmark column, the benchmark value decreases as the first exdividend date $t_1$ increases. Our formula successfully captures this phenomenon, while the other two approaches fail.

Next, we extend our comparison to the two-dividend case. The underlying stock is assumed to pay two dividends at year 0.5 and year 1 and the time to maturity is set to 1.5 years. Tables 6 and 7 illustrate the impacts of changing the initial stock price and the amount of dividend payments on the option value. Again, MAE and RMSE of Ours are also smaller than those of ContDiv and Model 1. Thus, we conclude that our pricing formulae can provide more accurate and consistent pricing results than other models.

Our option pricing model can be applied to extend the applicability of the first-passage model. A hypothetical example to analyse the impact of selling the firm’s asset to finance the repayment of one junior debt on the value of another unse-
Table 2. Comparing the effect of changing initial stock prices on pricing barrier calls with a single discrete dividend.

<table>
<thead>
<tr>
<th>S(0)</th>
<th>MC</th>
<th>L</th>
<th>Ours</th>
<th>error(MC)</th>
<th>error(L)</th>
<th>ContDiv</th>
<th>error(MC)</th>
<th>Model1</th>
<th>error(MC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>46</td>
<td>1.1265</td>
<td>1.1247</td>
<td>1.1260</td>
<td>0.0005</td>
<td>0.0013</td>
<td>1.1124</td>
<td>0.0141</td>
<td>1.1336</td>
<td>0.0071</td>
</tr>
<tr>
<td>48</td>
<td>1.3456</td>
<td>1.3416</td>
<td>1.3427</td>
<td>0.0029</td>
<td>0.0011</td>
<td>1.3317</td>
<td>0.0139</td>
<td>1.3641</td>
<td>0.0184</td>
</tr>
<tr>
<td>50</td>
<td>1.5054</td>
<td>1.5015</td>
<td>1.5026</td>
<td>0.0028</td>
<td>0.0011</td>
<td>1.4952</td>
<td>0.0102</td>
<td>1.5417</td>
<td>0.0363</td>
</tr>
<tr>
<td>52</td>
<td>1.5829</td>
<td>1.5785</td>
<td>1.5796</td>
<td>0.0033</td>
<td>0.0011</td>
<td>1.5767</td>
<td>0.0062</td>
<td>1.6401</td>
<td>0.0572</td>
</tr>
<tr>
<td>54</td>
<td>1.5661</td>
<td>1.5561</td>
<td>1.5571</td>
<td>0.0089</td>
<td>0.0010</td>
<td>1.5598</td>
<td>0.0063</td>
<td>1.6422</td>
<td>0.0762</td>
</tr>
<tr>
<td>56</td>
<td>1.4389</td>
<td>1.4299</td>
<td>1.4310</td>
<td>0.0079</td>
<td>0.0011</td>
<td>1.4395</td>
<td>0.0005</td>
<td>1.5423</td>
<td>0.1034</td>
</tr>
<tr>
<td>58</td>
<td>1.2112</td>
<td>1.2083</td>
<td>1.2093</td>
<td>0.0019</td>
<td>0.0010</td>
<td>1.2228</td>
<td>0.0116</td>
<td>1.3463</td>
<td>0.1352</td>
</tr>
<tr>
<td>60</td>
<td>0.9164</td>
<td>0.9097</td>
<td>0.9106</td>
<td>0.0059</td>
<td>0.0009</td>
<td>0.9266</td>
<td>0.0102</td>
<td>1.0700</td>
<td>0.1536</td>
</tr>
<tr>
<td>62</td>
<td>0.5667</td>
<td>0.5597</td>
<td>0.5602</td>
<td>0.0065</td>
<td>0.0005</td>
<td>0.5745</td>
<td>0.0078</td>
<td>0.7358</td>
<td>0.1691</td>
</tr>
<tr>
<td>64</td>
<td>0.1932</td>
<td>0.1865</td>
<td>0.1868</td>
<td>0.0065</td>
<td>0.0003</td>
<td>0.1932</td>
<td>0.0000</td>
<td>0.3697</td>
<td>0.1765</td>
</tr>
</tbody>
</table>

Notes: All other numerical settings are the same as those in figure 2 except that the initial stock prices are listed in the first column. MC and L denote the Monte Carlo simulation and the lattice method, respectively. ‘error(MC)’ (or ‘error(L)’) denote the absolute pricing error between each pricing formula and the Monte Carlo simulation (or the lattice method). MAE denotes the maximum absolute error and RMSE denotes the root-mean-squared error.

Table 3. Comparing the effect of changing the amount of the dividend payout on pricing barrier calls with a single discrete dividend.

<table>
<thead>
<tr>
<th>c1_1</th>
<th>Benchmark</th>
<th>Ours</th>
<th>error</th>
<th>ContDiv</th>
<th>error</th>
<th>Model1</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>1.5759</td>
<td>1.5730</td>
<td>0.0029</td>
<td>1.5705</td>
<td>0.0054</td>
<td>1.5857</td>
<td>0.0098</td>
</tr>
<tr>
<td>0.6</td>
<td>1.5438</td>
<td>1.5435</td>
<td>0.0003</td>
<td>1.5387</td>
<td>0.0051</td>
<td>1.5680</td>
<td>0.0242</td>
</tr>
<tr>
<td>0.9</td>
<td>1.5202</td>
<td>1.5129</td>
<td>0.0073</td>
<td>1.5062</td>
<td>0.0140</td>
<td>1.5486</td>
<td>0.0283</td>
</tr>
<tr>
<td>1.2</td>
<td>1.4868</td>
<td>1.4815</td>
<td>0.0053</td>
<td>1.4729</td>
<td>0.0139</td>
<td>1.5273</td>
<td>0.0405</td>
</tr>
<tr>
<td>1.5</td>
<td>1.4478</td>
<td>1.4493</td>
<td>0.0015</td>
<td>1.4390</td>
<td>0.0088</td>
<td>1.5044</td>
<td>0.0566</td>
</tr>
<tr>
<td>1.8</td>
<td>1.4147</td>
<td>1.4163</td>
<td>0.0017</td>
<td>1.4045</td>
<td>0.0102</td>
<td>1.4798</td>
<td>0.0652</td>
</tr>
<tr>
<td>2.1</td>
<td>1.3843</td>
<td>1.3828</td>
<td>0.0015</td>
<td>1.3694</td>
<td>0.0150</td>
<td>1.4538</td>
<td>0.0695</td>
</tr>
<tr>
<td>2.4</td>
<td>1.3459</td>
<td>1.3488</td>
<td>0.0030</td>
<td>1.3338</td>
<td>0.0121</td>
<td>1.4262</td>
<td>0.0804</td>
</tr>
</tbody>
</table>

Notes: All settings are the same as the settings in Table 2 except that the initial stock price is set as 50 and that the dividend c_1 is listed in the first column. Benchmark denotes the benchmark value generated by the Monte Carlo simulation. error denotes the absolute pricing error between each pricing formula and the Monte Carlo simulation.

Table 4. Comparing the effect of changing the stock price volatility on pricing barrier calls with a single discrete dividend.

<table>
<thead>
<tr>
<th>Volatility</th>
<th>Benchmark</th>
<th>Ours</th>
<th>error</th>
<th>ContDiv</th>
<th>error</th>
<th>Model1</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2.0707</td>
<td>2.0756</td>
<td>0.0049</td>
<td>2.0552</td>
<td>0.0154</td>
<td>2.0612</td>
<td>0.0094</td>
</tr>
<tr>
<td>0.2</td>
<td>1.5054</td>
<td>1.5026</td>
<td>0.0028</td>
<td>1.4952</td>
<td>0.0102</td>
<td>1.5417</td>
<td>0.0363</td>
</tr>
<tr>
<td>0.3</td>
<td>0.7215</td>
<td>0.7167</td>
<td>0.0047</td>
<td>0.7172</td>
<td>0.0043</td>
<td>0.7534</td>
<td>0.0320</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3625</td>
<td>0.3611</td>
<td>0.0014</td>
<td>0.3627</td>
<td>0.0002</td>
<td>0.3846</td>
<td>0.0221</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2035</td>
<td>0.1998</td>
<td>0.0037</td>
<td>0.2013</td>
<td>0.0022</td>
<td>0.2144</td>
<td>0.0109</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1205</td>
<td>0.1197</td>
<td>0.0007</td>
<td>0.1209</td>
<td>0.0004</td>
<td>0.1292</td>
<td>0.0087</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0767</td>
<td>0.0764</td>
<td>0.0003</td>
<td>0.0773</td>
<td>0.0006</td>
<td>0.0827</td>
<td>0.0060</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0526</td>
<td>0.0511</td>
<td>0.0014</td>
<td>0.0518</td>
<td>0.0007</td>
<td>0.0556</td>
<td>0.0030</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0366</td>
<td>0.0356</td>
<td>0.0010</td>
<td>0.0361</td>
<td>0.0005</td>
<td>0.0388</td>
<td>0.0021</td>
</tr>
<tr>
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<td>0.0255</td>
<td>0.0255</td>
<td>0.0000</td>
<td>0.0260</td>
<td>0.0005</td>
<td>0.0279</td>
<td>0.0024</td>
</tr>
</tbody>
</table>

Notes: All numerical settings are the same as those settings in table 2 except that the initial stock price is 50 and that the volatility of the stock price is listed in the first column.

cured senior debt that has an asset sale clause† is illustrated in figure 3. The firm is assumed to repay the former debt by the amount $150 at year 1.5, and we vary the maturity of the latter debt to analyse the effect of repaying the former debt on the value of the latter debt. Similar arguments have been studied empirically in Linn and Stock (2005). They find strong support for the following hypothesis: When the junior debt matures prior to the senior unsecured debt, the security of the senior unsecured debt is threatened and the default spread (of the senior debt) increases. One possible explanation is that the

†In section 1, we show that allowing the sale of the issuer’s asset to finance the loan repayments is much more common than putting restrictions on the asset sale as discussed in Eom et al. (2004) and Billett et al. (2007).
Table 5. Comparing the effect of changing the exdividend date on pricing barrier calls with a single discrete dividend.

<table>
<thead>
<tr>
<th>$t$</th>
<th>Benchmark</th>
<th>Ours</th>
<th>error</th>
<th>ContDiv</th>
<th>error</th>
<th>Model1</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.5425</td>
<td>1.5378</td>
<td>0.0047</td>
<td>1.4938</td>
<td>0.0486</td>
<td>1.5408</td>
<td>0.0016</td>
</tr>
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<td>0.0405</td>
<td>1.5410</td>
<td>0.0063</td>
</tr>
<tr>
<td>0.3</td>
<td>1.5291</td>
<td>1.5262</td>
<td>0.0029</td>
<td>1.4945</td>
<td>0.0346</td>
<td>1.5412</td>
<td>0.0121</td>
</tr>
<tr>
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<td>1.5236</td>
<td>1.5160</td>
<td>0.0076</td>
<td>1.4948</td>
<td>0.0287</td>
<td>1.5415</td>
<td>0.0179</td>
</tr>
<tr>
<td>0.5</td>
<td>1.5054</td>
<td>1.5026</td>
<td>0.0028</td>
<td>1.4952</td>
<td>0.0102</td>
<td>1.5417</td>
<td>0.0363</td>
</tr>
<tr>
<td>0.6</td>
<td>1.4903</td>
<td>1.4861</td>
<td>0.0042</td>
<td>1.4955</td>
<td>0.0053</td>
<td>1.5419</td>
<td>0.0516</td>
</tr>
<tr>
<td>0.7</td>
<td>1.4737</td>
<td>1.4658</td>
<td>0.0079</td>
<td>1.4958</td>
<td>0.0222</td>
<td>1.5421</td>
<td>0.0684</td>
</tr>
<tr>
<td>0.8</td>
<td>1.4391</td>
<td>1.4399</td>
<td>0.0007</td>
<td>1.4962</td>
<td>0.0571</td>
<td>1.5423</td>
<td>0.1032</td>
</tr>
<tr>
<td>0.9</td>
<td>1.4036</td>
<td>1.4029</td>
<td>0.0007</td>
<td>1.4965</td>
<td>0.0929</td>
<td>1.5425</td>
<td>0.1389</td>
</tr>
</tbody>
</table>

Notes: All numerical settings are the same as those settings in table 2, except that the initial stock price is 50 and the exdividend date is listed in the first column.

Table 6. Comparing the effect of changing initial stock prices on pricing barrier calls with two dividends.

<table>
<thead>
<tr>
<th>$S(0)$</th>
<th>Benchmark</th>
<th>Ours</th>
<th>error</th>
<th>ContDiv</th>
<th>error</th>
<th>Model1</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>46</td>
<td>0.9156</td>
<td>0.9122</td>
<td>0.0034</td>
<td>0.9003</td>
<td>0.0153</td>
<td>0.9493</td>
<td>0.0338</td>
</tr>
<tr>
<td>48</td>
<td>1.0033</td>
<td>1.0028</td>
<td>0.0005</td>
<td>0.9964</td>
<td>0.0069</td>
<td>1.0619</td>
<td>0.0586</td>
</tr>
<tr>
<td>50</td>
<td>1.0538</td>
<td>1.0493</td>
<td>0.0045</td>
<td>1.0481</td>
<td>0.0058</td>
<td>1.1322</td>
<td>0.0783</td>
</tr>
<tr>
<td>52</td>
<td>1.0484</td>
<td>1.0438</td>
<td>0.0046</td>
<td>1.0479</td>
<td>0.0005</td>
<td>1.1319</td>
<td>0.1035</td>
</tr>
<tr>
<td>54</td>
<td>0.9880</td>
<td>0.9843</td>
<td>0.0037</td>
<td>0.9934</td>
<td>0.0055</td>
<td>1.1178</td>
<td>0.1298</td>
</tr>
<tr>
<td>56</td>
<td>0.8771</td>
<td>0.8737</td>
<td>0.0035</td>
<td>0.8872</td>
<td>0.0101</td>
<td>1.0316</td>
<td>0.1545</td>
</tr>
<tr>
<td>58</td>
<td>0.7241</td>
<td>0.7192</td>
<td>0.0049</td>
<td>0.7357</td>
<td>0.0116</td>
<td>0.8990</td>
<td>0.1749</td>
</tr>
<tr>
<td>60</td>
<td>0.5364</td>
<td>0.5315</td>
<td>0.0049</td>
<td>0.5485</td>
<td>0.0122</td>
<td>0.7287</td>
<td>0.1924</td>
</tr>
<tr>
<td>62</td>
<td>0.3249</td>
<td>0.3233</td>
<td>0.0016</td>
<td>0.3371</td>
<td>0.0122</td>
<td>0.5317</td>
<td>0.2068</td>
</tr>
<tr>
<td>64</td>
<td>0.1104</td>
<td>0.1077</td>
<td>0.0052</td>
<td>0.1131</td>
<td>0.0027</td>
<td>0.3192</td>
<td>0.2088</td>
</tr>
</tbody>
</table>

Notes: All settings are the same as the settings in table 2 except that the underlying stock is assumed to pay a 1 dollar dividend at year 0.5 and year 1, and the time to maturity is 1.5 years.

Table 7. Comparing the effect of changing the amounts of dividends on pricing barrier calls with two dividends.

<table>
<thead>
<tr>
<th>$c_1 = c_2$</th>
<th>Benchmark</th>
<th>Ours</th>
<th>error</th>
<th>ContDiv</th>
<th>error</th>
<th>Model1</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>1.1305</td>
<td>1.1238</td>
<td>0.0067</td>
<td>1.1232</td>
<td>0.0073</td>
<td>1.1514</td>
<td>0.0210</td>
</tr>
<tr>
<td>0.6</td>
<td>1.0948</td>
<td>1.0933</td>
<td>0.0015</td>
<td>1.0923</td>
<td>0.0025</td>
<td>1.1462</td>
<td>0.0514</td>
</tr>
<tr>
<td>0.9</td>
<td>1.0585</td>
<td>1.0606</td>
<td>0.0021</td>
<td>1.0594</td>
<td>0.0010</td>
<td>1.1364</td>
<td>0.0780</td>
</tr>
<tr>
<td>1.2</td>
<td>1.0279</td>
<td>1.0269</td>
<td>0.0010</td>
<td>1.0248</td>
<td>0.0031</td>
<td>1.1222</td>
<td>0.0943</td>
</tr>
<tr>
<td>1.5</td>
<td>0.9864</td>
<td>0.9897</td>
<td>0.0033</td>
<td>0.9885</td>
<td>0.0021</td>
<td>1.1038</td>
<td>0.1174</td>
</tr>
<tr>
<td>1.8</td>
<td>0.9552</td>
<td>0.9535</td>
<td>0.0018</td>
<td>0.9508</td>
<td>0.0045</td>
<td>1.0812</td>
<td>0.1260</td>
</tr>
<tr>
<td>2.1</td>
<td>0.9147</td>
<td>0.9156</td>
<td>0.0009</td>
<td>0.9118</td>
<td>0.0030</td>
<td>1.0548</td>
<td>0.1401</td>
</tr>
<tr>
<td>2.4</td>
<td>0.8769</td>
<td>0.8772</td>
<td>0.0003</td>
<td>0.8715</td>
<td>0.0055</td>
<td>1.0247</td>
<td>0.1478</td>
</tr>
</tbody>
</table>

Notes: All settings are the same as the settings in table 6, except that the initial stock price is 50, and the underlying stock is assumed to pay dividend $c_1$ at year 0.5 and dividend $c_2$ at year 1. The amounts of the dividend payout are listed in the first column.

The repayment of the former debt may weaken the financial status of the issuing firm and increase the credit risk of the latter debt, if the former debt matures prior to the latter one. The prices of the latter debt generated by our formulae (marked in solid squares in figure 3) do catch this feature by generating a significant price drop from 2893.77 (with a time to maturity of 1.5 years) to 2885.52 (with a time to maturity of 1.52 years). On the other hand, many structural credit risk models (like Kim et al. (1993) and Longstaff and Schwartz (1995)) use a constant continuous payout ratio instead of a discrete payment. In this experiment, the continuous payout ratio used to approximate the discrete payout at year 1.5 is estimated by the formula proposed in Geske and Shastri (1985). Under the continuous payout setting, the pricing results for the latter debt (marked in solid triangles) simply decrease smoothly with the increment in the latter debt maturity to reflect the change of time value and can not precisely reflect the risk that the latter debt holders might suffer due to the repayment of the former debt.
6. Conclusions

Most stock dividends are paid discretely rather than continuously. However, no satisfactory analytical formulae for pricing barrier stock options with discrete dividends are announced. This paper provides accurate analytical formulae for pricing barrier stock options with discrete dividend payouts. Numerical results are given to confirm the superiority of our formulae over other analytical formulae. Our formulae can also extend the applicability of the first passage model, a popular credit risk model. The falls of the stock price due to the discrete dividend payments are analogous to selling the firm’s assets to finance the debt or dividend payments. Thus, our formulae can estimate how the firm’s repayments influence its financial status and the credit qualities of other outstanding debts.

Acknowledgements

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References


Appendix A: Reexpress the integration of exponential functions in terms of the CDF of a multi-variate normal distribution

The pricing formulae in this paper can be expressed in terms of the CDF of a standard normal distribution by reexpressing the vector of integrators. The proof for the Theorem is given as follows.

Proof

The multivariate integral in equation (22) can be reexpressed in terms of the exponent term of integrators: \( x_1, x_2, \ldots, x_n \) into the CDF of a multivariate standard normal distribution. For convenience, let \( x \equiv (x_1, x_2, \ldots, x_n) \) denote a column vector of \( n \) integrators. The proof for the Theorem is given as follows.

\[
\mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{B}^T \mathbf{x} + \mathbf{C} = -\frac{1}{2} \mathbf{y}^T \Sigma^{-1} \mathbf{y} + C.
\]  

where \( \mathbf{C} \) denotes a scalar that does not depend on \( \mathbf{x} \), \( \mathbf{y} \) denotes a vector and \( \mathbf{\Sigma} \) denotes a covariance matrix. The above equation and the values of \( \mathbf{C} \), \( \mathbf{y} \) and \( \mathbf{\Sigma} \) can be derived by the following lemma:

Lemma A.1 Under the premises that \( \mathbf{A} \) is a symmetric invertible \( n \times n \) matrix, and \( \mathbf{x}, \mathbf{B} \) are both \( n \times 1 \) vectors, we have

\[
\mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{B}^T \mathbf{x} = (\mathbf{x} + \frac{1}{2} \mathbf{A}^{-1} \mathbf{B})^T \mathbf{A} (\mathbf{x} + \frac{1}{2} \mathbf{A}^{-1} \mathbf{B}) - \frac{1}{4} \mathbf{B}^T \mathbf{A}^{-1} \mathbf{B}.
\]
Proof By expanding the right-hand side of equation (46), we have

\[
\begin{align*}
 ( x + \frac{1}{2} A^{-1} B )^T A ( x + \frac{1}{2} A^{-1} B ) & = \frac{1}{4} B^T A^{-1} B \\
= x^T A x + \frac{1}{2} B^T ( A^{-1})^T A x + \frac{1}{2} x^T A A^{-1} B \\
+ \frac{1}{4} B^T ( A^{-1} )^T A A^{-1} B - \frac{1}{4} B^T A^{-1} B \\
= x^T A x + \frac{1}{2} B^T x + \frac{1}{2} x^T A x - \frac{1}{4} B^T A^{-1} B - \frac{1}{4} B^T A^{-1} B \\
= x^T A x + B^T x + B^T x - \frac{1}{4} B^T A^{-1} B \\
\text{where equation (52) is substituted into the right-hand side of equation (46).}
\end{align*}
\]

\[
\begin{align*}
\text{(12)}
\end{align*}
\]

where the equation \((A^{-1})^T = -A^{-1}\) due to the symmetry of \(A\) is substituted into equation (47). Since \(B^T x\) is a scalar, we have \(B^T x = (B^T x)^T = x^T B\). Equation (48) is obtained by substituting the aforementioned equalities into equation (47).

By applying Lemma A.1 to equation (45), we obtain

\[
\begin{align*}
\sqrt{\Sigma} y & = S^{-1} ( x - m ) + \Sigma
\end{align*}
\]

where \(S\) denotes a diagonal matrix. To solve \(S\), we first subtract \(C\) from both sides of equation (51) to yield

\[
\begin{align*}
( x - m )^T A ( x - m ) & = -\frac{1}{2} \Sigma^{-1} y \\
\text{where equation (52) is substituted into the right-hand side of equation (53). (} S^{-1} )^T \text{ is equal to } S^{-1} \text{ due to the symmetry of } S \text{ and this equation is substituted into equation (54). By comparing the left-hand side of equation (53) and equation (55), we have } \frac{1}{2} (S \Sigma S)^{-1} = A, \text{ which can be rewritten as } S \Sigma S = (2 \Sigma A)^{-1}. \text{ Recall that } S \text{ is a diagonal matrix. All diagonal elements of } \Sigma \text{ are 1 since } \Sigma \text{ is a covariance matrix of multivariate standard normal random variables. Thus, we have } (S \Sigma S)_{i,j} = S_{i,j}, \text{ which leads us to obtain}
\end{align*}
\]

\[
\begin{align*}
S_{i,j} & = \left\{ \begin{array}{ll}
\sqrt{((2 \Sigma A)^{-1})_{i,j}} & \text{if } i = j \\
0 & \text{otherwise}
\end{array} \right.
\end{align*}
\]

and \(\Sigma \equiv (-2 SAS)^{-1}\).

Now we can express equation (22) in terms of \(C, m, S, \text{ and } \Sigma\) defined above. By applying the change of variable defined in equation (52), equation (22) can be rewritten as

\[
\begin{align*}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{x^T A x + B^T x + C} dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(x - m)^T A (x - m) + C} dx dy.
\end{align*}
\]

Since the elements in vector \(y\) can be represented as \((y_{S_{1,1}}, y_{S_{2,2}}, \ldots, y_{S_{n,n}})^T\), the Jacobian determinant can be straightforwardly computed to obtain \(\frac{dy}{dy} = \prod_{i=1}^{n} S_{i,i} = |S|\). Thus, equation (56) can be further rewritten as the following closed-form formula:

\[
\begin{align*}
\frac{e^{C}}{|S|} \int_{-\infty}^{p - m_{1}} \int_{-\infty}^{p - m_{2}} \cdots \int_{-\infty}^{p - m_{n}} e^{-\frac{1}{2} y^T \Sigma^{-1} y} dy = \frac{e^{C}}{|\sqrt{\Sigma}|} \int_{-\infty}^{p - m_{1}} \int_{-\infty}^{p - m_{2}} \cdots \int_{-\infty}^{p - m_{n}} e^{-\frac{1}{2} y^T \Sigma^{-1} y} dy
\end{align*}
\]

\[
\begin{align*}
\text{where } |S| \sqrt{|\Sigma|} = \sqrt{|S \Sigma S|} = \sqrt{|-2A|^{-1}}, \text{ and } |S| = 2^n = |S| \text{ are substituted into equation (57).}
\end{align*}
\]

Appendix B: Proof of Corollary 3.1

Corollary 4.1 can be derived from Theorem 3.1 by setting \(n = 2\) as follows:

First, to make \(a_1 x^2 + a_2 x y + a_3 y^2 + a_4 x + a_5 y + a_6\) equal \(x^T A x + x^T B + C\), we set

\[
\begin{align*}
A & \equiv \left( \frac{a_1}{a_2} \frac{a_3}{a_5} \right), B \equiv \left( \frac{a_4}{a_6} \right), C \equiv a_6.
\end{align*}
\]

By substituting the above equations into Theorem 3.1, we have

\[
\begin{align*}
m & = -\frac{1}{2} A^{-1} B = -\frac{1}{4 a_1 a_3 - a_2^2} \left( 2a_3 a_4 - a_2 a_5 - a_2 a_6 \right) \\
C & = C - \frac{1}{4} B^T A^{-1} B = a_6 + \frac{2a_3 a_4 a_5 - a_2 a_5^2 - a_1 a_5 a_6}{4a_1 a_3 - a_2^2} \\
S_{1,1} & = \sqrt{\left((-2A)^{-1}\right)_{1,1}} = \frac{4a_1 a_3 - a_2^2}{\sqrt{4a_1 a_3 - a_2^2}} \\
S_{2,2} & = \sqrt{\left((-2A)^{-1}\right)_{2,2}} = \frac{4a_1 a_3 - a_2^2}{\sqrt{4a_1 a_3 - a_2^2}} \\
\Sigma & = (-2 SAS)^{-1} = \frac{1}{\sqrt{4a_1 a_3 - a_2^2}} \left( \frac{a_2}{\sqrt{4a_1 a_3 - a_2^2}} \right)
\end{align*}
\]

By substituting the above into equation (23), we obtain

\[
\begin{align*}
\int_{-\infty}^{p} \int_{-\infty}^{p} \exp\left( a_1 x^2 + a_2 x y + a_3 y^2 + a_4 x + a_5 y + a_6 \right) dx dy = e^{C} \sqrt{\Delta} \int_{-\infty}^{p - m_{1}} \int_{-\infty}^{p - m_{2}} \cdots \int_{-\infty}^{p - m_{n}} \exp\left( a_6 + \frac{a_2 a_4 a_5 - a_2^2 a_5 - a_1 a_5 a_6}{\sqrt{4a_1 a_3 - a_2^2}} \right) dy.
\end{align*}
\]

\[
\begin{align*}
= \frac{2\pi}{\sqrt{4a_1 a_3 - a_2^2}} \exp\left( a_6 + \frac{a_2 a_4 a_5 - a_2^2 a_5 - a_1 a_5 a_6}{\sqrt{4a_1 a_3 - a_2^2}} \right) \\
\text{where } \Delta = 4a_1 a_3 - a_2^2.
\end{align*}
\]
The remaining parameters are given by the following table:

<table>
<thead>
<tr>
<th>i</th>
<th>(a(i))</th>
<th>(a_1(i))</th>
<th>(a_2(i))</th>
<th>(a_3(i))</th>
<th>(a_4(i))</th>
<th>(a_5(i))</th>
<th>(a_6(i))</th>
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</thead>
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<td>(\alpha - ak_2)</td>
<td>(\alpha)</td>
<td>(\alpha - ak_2)</td>
<td>(\alpha)</td>
<td>(\alpha - ak_2)</td>
</tr>
<tr>
<td>2</td>
<td>(\alpha)</td>
<td>(\alpha - a_k)</td>
<td>(\alpha - ak_2)</td>
<td>(\alpha)</td>
<td>(\alpha - ak_2)</td>
<td>(\alpha)</td>
<td>(\alpha - ak_2)</td>
</tr>
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<td>(\frac{2k}{T - t_1} + \alpha)</td>
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<td>(\alpha - ak_1)</td>
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<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
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</tr>
<tr>
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<td>(\alpha + k_2\left(\frac{2k}{T - t_1} - \alpha\right))</td>
<td>(\alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(\alpha)</td>
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<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
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</tr>
<tr>
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</tr>
<tr>
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<td></td>
</tr>
<tr>
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<td>(\alpha - ak_1)</td>
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<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
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</tr>
<tr>
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<td>(\alpha + \alpha)</td>
<td>(\alpha - ak_2)</td>
<td>(\alpha - ak_1)</td>
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<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
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</tr>
<tr>
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<td>(\alpha - ak_1)</td>
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</tr>
<tr>
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<td>(\alpha - ak_1)</td>
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<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>(\alpha + \alpha)</td>
<td>(\alpha - ak_2)</td>
<td>(\alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>(\alpha + \alpha)</td>
<td>(\alpha - ak_2)</td>
<td>(\alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>(\frac{2k}{T - t_1} + \alpha + \alpha)</td>
<td>(\alpha - ak_2)</td>
<td>(\alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td></td>
</tr>
<tr>
<td>16</td>
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<td>(\alpha - ak_2)</td>
<td>(\alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td>(\frac{2k}{T - t_1} + \alpha - ak_1)</td>
<td></td>
</tr>
</tbody>
</table>

**Appendix C: Coefficients of the exponential terms of \(J(i)\)**

The coefficients \(a_1(i), a_2(i), a_3(i), a_4(i), a_5(i)\) and \(a_6(i)\) are defined by the following formulae:

\[
a_1(i) = -\frac{1}{2(T-t_2)},
\]
\[
a_2(i) = -\left(\frac{k_i^2}{2(T-t_3)} + \frac{1}{2(t_2-t_1)}\right),
\]
\[
a_3(i) = -\left(\frac{k_i^2}{2(t_2-t_1)} + \frac{1}{2t_1}\right),
\]
\[
a_6(i) = 0,
\]
\[
a_4(i) = \begin{cases} \frac{k_i^2}{T-t_2} & \text{if } i = 1, 2, 5, 6, 9, 10, 13, 14 \\ \frac{k_i^2}{T-t_2} & \text{otherwise} \end{cases}
\]
\[
a_5(i) = \begin{cases} \frac{k_i^2}{T-t_2} & \text{if } i = 1, 2, 3, 4, 9, 10, 11, 12 \\ \frac{k_i^2}{T-t_2} & \text{otherwise} \end{cases}
\]

The remaining parameters are given by the following table: