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Distributional and inferential properties of the process accuracy and process precision indices

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DISTRIBUTIONAL AND INFERENTIAL PROPERTIES OF THE PROCESS ACCURACY AND PROCESS PRECISION INDICES

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ABSTRACT

Process capability indices such as $C_p$, $k$, and $C_{pk}$, have been widely used in manufacturing industry to provide numerical measures on process potential and performance. While $C_p$ measures overall process variation, $k$ measures the degree of process departure. In this paper, we consider the index $C_p$ and a transformation of $k$ defined as $C_a = 1 - k$ which measures the degree of process centering. We refer to $C_p$ as the process precision index, and $C_a$ as the process accuracy index. We consider the estimators of $C_p$ and $C_a$, and investigate their statistical properties.
properties. For $C_p$, we obtain the UMVUE and the MLE. We show that this
UMVUE is consistent, and asymptotically efficient. For $C_p$, we investigate its
natural estimator. We obtain the first two moments of this estimator, and show
that the natural estimator is the MLE, which is asymptotically unbiased and
asymptotically efficient. We also propose an efficient test based on the UMVUE
of $C_p$. We show that the proposed test is the UMP test.

1. INTRODUCTION

Process capability indices, which establish the relationships between the
actual process performance and the manufacturing specifications (including the
target value and specification limits), have been the focus of recent research in
quality assurance and process capability (quality) analysis. Those capability
indices, quantifying process potential and performance, are important for any
successful quality improvement activities and quality program implementation.
Several indices widely used in manufacturing industry providing numerical
measures on whether a process meets the preset quality requirement, include $C_p$, $k$, and $C_{pk}$ which are defined as the following (see Kane (1986)):

$$C_p = \frac{USL - LSL}{6\sigma},$$

$$k = \frac{|\mu - m|}{d},$$

$$C_{pk} = \min \left\{ \frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma} \right\},$$

where $USL$ and $LSL$ are the upper and the lower specification limits preset by the
process engineers or product designers, $\mu$ is the process mean, $\sigma$ is the process
standard deviation, $m$ is the mid-point between the upper and the lower
specification limits ($m = (USL + LSL)/2$), and $d$ is half length of the specification
interval ($d = (USL - LSL)/2$). We have assumed the target value $T = m$ (which is
quite common in practical situations) for simplicity of our discussions.

The index $C_p$ was designed to measure the magnitude of the overall process
variation. For processes with two-sided specification limits, the percentage of
nonconforming items ($\%NC$) can be calculated as $1 - F(USL) + F(LSL)$, where
$F(\cdot)$ is the cumulative distribution function of the process characteristic $X$. On the
assumption of normality, $\%NC$ can be expressed as:

$$\%NC = 1 - \Phi \left[ \frac{USL - \mu}{\sigma} \right] + \Phi \left[ \frac{LSL - \mu}{\sigma} \right],$$
where $\Phi(\cdot)$ is the cumulative function of the standard normal distribution. If the process is perfectly centered, then $%NC$ can be expressed alternatively as $%NC = 2 - 2\Phi(3C_p)$. For example, $C_p = 1.00$ corresponds to $%NC = 2700$ ppm, and $C_p = 1.33$ corresponds to $%NC = 63$ ppm. Thus, the index $C_p$ provides an exact measure of the actual process yield. Since $C_p$ measures the magnitude of process variation, $C_p$ may be viewed as a process precision index.

While the precision index $C_p$ measures the magnitude of process variation, the index $k$ measures the departure of process mean, $\mu$, from the center-point $m$. Therefore, the transformation of $k$ defined as $C_a = 1 - k$ measures the degree of process centering (the ability to cluster around the center), which can be regarded as a process accuracy index. For example, $C_a = 1$ indicates that the process is perfectly centered ($\mu = m$), $C_a > 1/2$ indicates that $\mu$ is within half of the specification interval, and $C_a = 0$ indicates that $\mu$ is on the specification limits ($\mu = USL$, or $\mu = LSL$). On the other hand, if $C_a < 0$ then it indicates that $\mu$ falls outside the specification limits ($\mu > USL$, or $\mu < LSL$). Obviously, the process is severely off-center and it needs an immediate troubleshooting.

2. ESTIMATION OF $C_p$

To estimate the precision index $C_p$, we consider the natural estimator $\hat{C}_p$ defined as the following, where $S^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n - 1}$ is the conventional estimator of the process standard deviation $\sigma$, which may be obtained from a stable process.

$$\hat{C}_p = \frac{USL - LSL}{6S}.$$ 

The natural estimator $\hat{C}_p$ can be alternatively written as:

$$\hat{C}_p = (n - 1)^{1/2} \frac{(USL - LSL)}{6\sigma} \left[ (n - 1) S^2 \right]^{1/2} = (n - 1)^{1/2} C_p \left[ \frac{(n - 1) S^2}{\sigma^2} \right]^{1/2}.$$ 

On the assumption of normality, the statistic $(n - 1)S^2 / \sigma^2$ is distributed as $\chi^2_{n-1}$, a chi-square with $n - 1$ degrees of freedom. Therefore, the probability density function of $\hat{C}_p$ can be expressed as (Chou and Owen (1989)):

$$f(x) = 2 \frac{\Gamma[(n - 1)/2]}{\Gamma[n - 1/2]} \left[ \frac{(n - 1) C_p}{2} \right]^{(n - 1)/2} \exp\left[-(n - 1)\left(C_p(2x^2)^{-1}\right)\right],$$

for $x > 0$. The $r$-th moment of $\hat{C}_p$, therefore can be calculated as the following:
and the first two moments as well as the variance may be obtained as (see also Chou and Owen (1989), and Pearn, Kotz and Johnson (1992)):

\[
E(\hat{C}_p) = \frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \frac{(n-1)C_p^2}{2}.
\]

\[
E(\hat{C}_p^2) = \frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \frac{(n-1)C_p^2}{2} = \frac{n-1}{n-3} C_p^2.
\]

\[
\text{Var}(\hat{C}_p) = \frac{(n-1)}{2n-3} \frac{n-1}{2} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)^2} C_p^2.
\]

It can be shown that the coefficient of \(E(\hat{C}_p)/(n - 1)/2\) \(\Gamma(n - 2)/2/\Gamma(n - 1)/2 > 1\) for all \(n\). For \(n \geq 15\), this coefficient can be accurately approximated by \((4n - 4)/(4n - 7)\). Therefore, the natural estimator \(\hat{C}_p\) is biased, which overestimates the actual value of \(C_p\). Table 1 displays the values of \(E(\hat{C}_p)\) under the condition \(C_p = 1\) for various sample sizes \(n\). For the percentage bias to be less than one percent (\(|E(\hat{C}_p) - C_p|/C_p < 0.01\)), it requires the sample size \(n > 80\).

By setting

\[
b_j = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)} \left(\frac{n-1}{2}\right)^{1/2},
\]

we may obtain an unbiased estimator \(\hat{C}_p = b_j \hat{C}_p\). That is, \(E(\hat{C}_p) = C_p\). Since \(b_j < 1\), then the variance of \(\hat{C}_p\) is smaller than that of the natural estimator \(\hat{C}_p\). That is, \(\text{Var}(\hat{C}_p) < \text{Var}(\hat{C}_p)\). In the following, we investigate the statistical properties of \(\hat{C}_p\). We show that \(\hat{C}_p\) is the UMVUE of \(C_p\), which is consistent and asymptotically efficient.

An estimator \(\hat{\theta}_n\) of \(\theta\) is said to be consistent if for all \(\epsilon > 0\), \(p(\hat{\theta}_n - \theta \mid \epsilon) \rightarrow 0\) as \(n \rightarrow \infty\) for all \(\theta\). A sufficient condition for the consistency is that \(E(\hat{\theta}_n) \rightarrow \theta\) and \(\text{Var}(\hat{\theta}_n) \rightarrow 0\). Under regular conditions, the estimator \(\hat{\theta}_n\) is said to be asymptotically efficient if \(\hat{\theta}_n\) is asymptotically normal, \(n^{1/2} (\hat{\theta}_n - \theta) \rightarrow 0\), and \(\{n \cdot \text{Var}(\hat{\theta}_n - E(\hat{\theta}_n))\}\) converges to the Cramer-Rao Bound.
Table 1. Values of $E(\tilde{C}_p)$ corresponding to $C_p = 1$ for various sample sizes $n$. 

<table>
<thead>
<tr>
<th>Sample size</th>
<th>$E(\tilde{C}_p)$</th>
<th>Sample size</th>
<th>$E(\tilde{C}_p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
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</tr>
<tr>
<td>90</td>
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<td>1.000</td>
</tr>
</tbody>
</table>

Theorem 1. If the process characteristic follows normal distribution, then

(a) $\tilde{C}_p$ is the UMVUE of $C_p$.
(b) $\tilde{C}_p$ is consistent.
(c) \( n^{1/2} (\tilde{C}_p - C_p) \) converges to $N(0, [C_p]^2/2)$ in distribution.
(d) $\tilde{C}_p$ is asymptotically efficient.

Proof: (a) We first note that the statistic $\tilde{X}, \tilde{S}^2$ is sufficient and complete for $(\mu, \sigma^2)$. Since $E(\tilde{C}_p) = C_p$, and $\tilde{C}_p$ is a function of $(\tilde{X}, \tilde{S}^2)$ only, then by Lehmann-Scheffe' Theorem (Arnold (1990)) $\tilde{C}_p$ is the UMVUE of $C_p$.

(b) For all $\varepsilon > 0$, $p(\tilde{C}_p - C_p > \varepsilon) < E(\tilde{C}_p - C_p)^2 / \varepsilon^2$. Now, $E(\tilde{C}_p - C_p)^2 = \text{Var}(\tilde{C}_p) = E(\tilde{C}_p)^2 - C_p^2$. By Stirling's formula, we can show that $E(\tilde{C}_p)^2$ converges to $C_p^2$. Hence, $E(\tilde{C}_p - C_p)^2$ converges to zero. Therefore, $\tilde{C}_p$ converges to $C_p$ in probability and $\tilde{C}_p$ must be consistent.

(c) If the process characteristic is normally distributed, then it is clear that the statistic $n^{1/2} (\tilde{S}^2 - \sigma^2)$ converges to $N(0, 2\sigma^4)$ in distribution. We apply Cramer-$\delta$ Theorem (Arnold (1990)) with $g(t)$ defined as $g(t) = d/(3t^{1/2})$. Since $g(t) = -d/(6t^{1/2})$, and $[g(\sigma^2)]^2 = (C_p)^2/(4\sigma^4)$, then $n^{1/2} [g(\tilde{S}^2) - g(\sigma^2)] = n^{1/2} [d/(3\tilde{S}) - d/(3\sigma)] = n^{1/2} (\tilde{C}_p - C_p)$ converges to $N(0, 2\sigma^4 [g(\sigma^2)])$, or $N(0, C_p^2/2)$ in distribution. By (b) $\tilde{C}_p$ converges to $C_p$ in probability, then by Slutsky's Theorem (Arnold (1990)) $n^{1/2} (\tilde{C}_p - C_p)$ converges to $N(0, C_p^2/2)$, and so $n^{1/2} (\tilde{C}_p - C_p)$ converges to $N(0, [C_p]^2/2)$ in distribution.
If the knowledge on whether \( p(\mu \geq m) = 1 \), or 0 is available, then we can consider the estimator \( \tilde{C}_a = 1 - (X - m) \text{sgn}(\mu - m)/d \), where \( \text{sgn}(\mu - m) = 1 \) if \( \mu - m \geq 0 \), and \( \text{sgn}(\mu - m) = -1 \) if \( \mu - m < 0 \). Thus, \( \tilde{C}_a = 1 - (X - m)/d \) if \( \mu \geq m \), and \( \tilde{C}_a = 1 - (m - X)/d \) if \( \mu < m \). We can show that the estimator \( \tilde{C}_a \) is the MLE, and the UMVUE of \( C_a \). We can also show that the estimator \( \tilde{C}_a \) is consistent, and efficient.

Theorem 3. If the process characteristic follows normal distribution, then

(a) \( \tilde{C}_a \) is the MLE of \( C_a \).
(b) \( \tilde{C}_a \) is the UMVUE of \( C_a \).
(c) \( \tilde{C}_a \) is consistent.
(d) \( n^{1/2}(\tilde{C}_a - C_a) \) converges to \( N(0, 1/(3C_p^2)) \) in distribution.

Proof: (a) We first note that the statistic \((\tilde{\mu}, [(n - 1)/n]S^2)\) is the MLE of \((\mu, \sigma^2)\). By the invariance property of the MLE, \( \tilde{\mu} \) is the MLE of \( \mu \), and \( \tilde{C}_a \) is the MLE of \( C_a \).

(b) From Theorem 2(d), the Cramer-Rao bound = \( 1/[9n C_p^2] \). Since \( \tilde{C}_a \) is distributed as \( N(C_a, 1/[9nC_p^2]) \), then \( \tilde{C}_a \) is efficient for \( C_a \), and is the UMVUE of \( C_a \).

(c) For all \( \epsilon > 0 \), \( p(\tilde{C}_a - C_a > \epsilon) < E(\tilde{C}_a - C_a)^2/\epsilon^2 \). Now, \( E(\tilde{C}_a - C_a)^2 = \text{Var}(\tilde{C}_a) = 1/[9nC_p^2] \) converges to zero, and so \( \tilde{C}_a \) must be consistent.

(d) From (c), \( \tilde{C}_a \) converges to \( C_a \) in probability. From Theorem 2(c), \( n^{1/2}(\tilde{C}_a - C_a) \) converges to \( N(0, 1/[3C_p^2]) \) in distribution. By Slutsky's Theorem (Arnold (1990)), \( n^{1/2}(\tilde{C}_a - C_a) \) converges to \( N(0, 1/[3C_p^2]) \) in distribution.

In fact, since \( \tilde{X} \) and \( S^2 \) are mutually independent, then \( \tilde{C}_a \) and \( \tilde{C}_p \) are also mutually independent. Therefore, since \( Z = 3n \tilde{C}_p(\tilde{C}_a - C_a) \) is distributed as \( N(0, 1) \) and \( W = (n - 1)(b/C_p^2)/[\tilde{C}_p] \) is distributed as \( \chi^2_{n-1} \), then \( 3n^{1/2}(\tilde{C}_a - C_a)/b = Z/[W/(n-1)^{1/2}] \) is distributed as \( t_{n-1} \), a t distribution with \( n - 1 \) degrees of freedom. Therefore, the \( \alpha \)-level confidence interval for \( C_a \) can be established as:

\[
\left[ \tilde{C}_a - \frac{b_1 f_{n-1, \alpha/2}}{3 \sqrt{n} \tilde{C}_p}, \tilde{C}_a + \frac{b_1 f_{n-1, \alpha/2}}{3 \sqrt{n} \tilde{C}_p} \right].
\]

where \( f_{n-1, \alpha} \) is the upper \( \alpha \)-th quantile of the \( f_{n-1} \) distribution. The length, \( L \), of the confidence interval, therefore is
Table 4. E(\(\ell\)) for \(C_a\) with \(C_p = 1\) and \(\alpha = 0.05\).

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>E((\ell))</th>
<th>Sample Size</th>
<th>E((\ell))</th>
</tr>
</thead>
<tbody>
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<td>10</td>
<td>0.464</td>
<td>120</td>
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<td>0.247</td>
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<td>0.212</td>
<td>180</td>
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</tr>
<tr>
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<td>0.189</td>
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<tr>
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<td>0.171</td>
<td>220</td>
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</tr>
<tr>
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</tr>
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</tr>
<tr>
<td>100</td>
<td>0.132</td>
<td>300</td>
<td>0.076</td>
</tr>
</tbody>
</table>

\[
\ell = 2 \times \frac{b_f t_{n-1, \alpha/2}}{\sqrt{n} C_p}.
\]

The expected value \(E(\ell)\) and the variance \(\text{Var}(\ell)\) of the length of the confidence interval \(\ell\), can be found as:

\[
E(\ell) = 2 \left( \frac{t_{n-1, \alpha/2}}{\sqrt{n}} \right) \sqrt{\frac{2}{n - 1}} \frac{\Gamma[n/2]}{\Gamma((n - 1)/2)} \frac{1}{C_p},
\]

\[
\text{Var}(\ell) = 4 \left( \frac{t_{n-1, \alpha/2}}{3 \sqrt{n}} \right)^2 \left\{ 1 - \left[ 1 - \sqrt{\frac{2}{n - 1}} \frac{\Gamma[n/2]}{\Gamma((n - 1)/2)} \right]^2 \right\} \frac{1}{(C_p)^2},
\]

for given sample size \(n\). Table 4 displays the expected lengths, \(E(\ell)\), of the \(\alpha\)-level confidence intervals for the accuracy index \(C_a\) with \(C_p = 1\) and \(\alpha = 0.05\) for various sample sizes \(n\).

### 4. Tests for Process Capability

To judge whether a given process meets the preset capability requirement and runs under the desired quality condition. We can consider the following statistical testing hypothesis for \(C_p\): \(H_0: C_p \leq C\), and \(H_1: C_p > C\). Process fails to meet the capability (quality) requirement if \(C_p \leq C\), and meets the capability requirement if \(C_p > C\). We define the test \(\phi^*(x)\) as: \(\phi^*(x) = 1\) if \(\bar{C}_p > c_0\), and \(\phi^*(x) = 0\) otherwise. Thus, the test \(\phi^*\) rejects the null hypothesis \(H_0\) (\(C_p \leq C\)) if \(\bar{C}_p > c_0\).
Table 5. Critical values $c_0$ for $C = 1.00$ with $n = 10(10)100$, and $\alpha = 0.01, 0.025, 0.05$.

<table>
<thead>
<tr>
<th>Sample size</th>
<th>0.01</th>
<th>0.025</th>
<th>0.05</th>
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</thead>
<tbody>
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<td>1.204</td>
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<tr>
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<td>1.230</td>
<td>1.187</td>
<td>1.152</td>
</tr>
<tr>
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<td>1.141</td>
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<tr>
<td>90</td>
<td>1.198</td>
<td>1.162</td>
<td>1.132</td>
</tr>
<tr>
<td>100</td>
<td>1.187</td>
<td>1.153</td>
<td>1.125</td>
</tr>
</tbody>
</table>

with type I error $\alpha(c_0) = \alpha$, the chance of incorrectly judging an incapable process $(C_p \leq C)$ as capable $(C_p > C)$. The critical value, $c_0$, can be determined as:

\[
p[c_p > c_0 | C_p = C] = \alpha
\]

\[
p\left[ (b_i \sqrt{n-1} C_p) (K)^{-1/2} > c_0 | C_p = C \right] = \alpha
\]

\[
p\left[ K < \left( b_i \right)^2 (n - 1) \left( \frac{C_p}{c_0} \right)^2 \right] = \alpha
\]

Hence, we have

\[
(b_i)^2 (n - 1) \left( \frac{C_p}{c_0} \right)^2 = \chi^2_{(\alpha, n-1)}
\]

where $\chi^2_{(\alpha, n-1)}$ is the lower $\alpha$-th quantile of $\chi^2_{n-1}$ distribution, or,

\[
c_0 = \frac{b_i \sqrt{n-1} C}{\sqrt{\chi^2_{(\alpha, n-1)}}}
\]

Therefore, if $c_p > c_0$, then $\phi^*(x) = 1$ and we reject the null hypothesis $H_0$ and conclude that the process meets the capability requirement ($C_p > C$). Otherwise, we can not conclude that the process meets the capability requirement. Tables 5 displays the critical values $c_0$ for $C = 1.00$ with sample sizes $n = 10(10)100$, and $\alpha$-risk $= 0.01, 0.025, 0.05$ (the chance of incorrectly concluding a process with $C_p \leq C$ as one with $C_p > C$).
Theorem 4. For the testing hypothesis $H_0$: $C_p \leq C$ and $H_1$: $C_p > C$, the test defined as $\phi^*(x) = 1$ if $C_p > c_0$, and $\phi^*(x) = 0$ otherwise, is the UMP test of level $\alpha$, where $c_0$ is determined by $E_c[\phi^*(x)] = \alpha$.

Proof: For the test, the power function is:

$$\beta(C_p, \phi^*) = E_{C_p}[\phi^*(x)] = P_{C_p}[\chi^2_{n-1} < \frac{(n-1)b^2C_p^2}{c_0^2}] .$$

For $\alpha(c_0) = \alpha$, $c_0 = \frac{C}{b} \sqrt{n-1}$, where $\chi^2_{n-1, 1-\alpha}$ satisfies

$$P[\chi^2_{n-1} > \chi^2_{n-1, 1-\alpha}] = 1 - \alpha.$$ 

Since for $C_p > C > 0$, $\frac{f_{C_p}(x', C_p)}{f_{C_p}(x, C_p)} > \frac{f_{C_p}(x, C_p)}{f_{C_p}(x, C_p)}$ if and only if $x' > x > 0$,

then $\{f_{C_p}(x, C_p)\} C_p > 0$ has MLR (monotone likelihood ratio) property in $C_p$. Therefore, the test $\phi^*$ must be the UMP test.

5. CONCLUSIONS

Process capability indices such as $C_p$, $k$, and $C_{pk}$, have been widely used in manufacturing industry to provide numerical measures on process potential and performance. The index $C_p$ measures the overall process variation, and the index $k$ measures the degree of process departure. In this paper, we considered $C_p$ and a transformation of $k$ defined as $C_a = 1 - k$. We referred to $C_p$ as the process precision index, and $C_a$ as the process accuracy index which measures the degree of process centering.

We considered the estimators of $C_p$ and $C_a$ and investigated their statistical properties. For $C_p$, we obtained the UMVUE and the MLE. We showed that this UMVUE is consistent, and asymptotically efficient. For $C_a$, we investigated its natural estimator. We showed that this natural estimator is the MLE, which is asymptotically unbiased and asymptotically efficient. In addition, we proposed an efficient test based on the UMVUE of $C_p$. Using this test, the practitioners can judge whether their processes meet the capability requirement preset in the factory. We showed that the proposed test is in fact the UMP test.
Theorem 1: The probability density function of $\tilde{C}_s$ can be expressed as:
\[
 f(x) = 6 C_p \sqrt{\frac{n}{2\pi}} \cosh \left\{ 9n \left[ C_p \right]^2 k \left( 1 - x \right) \right\} \exp \left\{ - \frac{9n \left[ C_p \right]^2 k \left( 1 - x \right)^2} {2} \right\}.
\]

Proof: For $-\infty < x \leq 1$, the probability density function $f(x)$ is
\[
 f(x) = \frac{d}{dx} p(\tilde{C}_a \leq x) = \frac{d}{dx} p(1 - \tilde{k} \leq x) = \frac{d}{dx} \left( 1 - p(\tilde{k} \leq 1 - x) \right)
\]
\[
= g(1 - x), \quad \text{where } g(x) \text{ is the probability density function of } \tilde{k}.
\]

Now, the statistic $Y = \frac{\bar{X} - m}{d}$ is distributed as $N \left( \mu - \frac{m}{d}, \left[ 9n \left( C_p \right)^2 \right]^{-1} \right)$, a normal distribution with mean $\mu_Y = (\mu - m)/d$ and variance $\sigma_Y^2 = \left[ 9n \left( C_p \right)^2 \right]^{-1}$.

Since $\tilde{k} = |Y| = \frac{\bar{X} - m}{d}$ has a folded normal distribution, then the probability density function of $\tilde{k}$ is:
\[
g(y) = \phi(y) + \phi(-y)
\]
\[
= \frac{1}{\sqrt{2\pi} \sigma_Y} \exp \left\{ - \frac{y - \mu_Y}{2(\sigma_Y)^2} \right\} + \exp \left\{ - \frac{y + \mu_Y}{2(\sigma_Y)^2} \right\}
\]
\[
= \frac{1}{\sqrt{2\pi} \sigma_Y} \exp \left\{ - \frac{y^2 + (\mu_Y)^2}{2(\sigma_Y)^2} \right\} \left\{ \exp \left[ \frac{y \times \mu_Y}{(\sigma_Y)^2} \right] + \exp \left[ - \frac{y \times \mu_Y}{(\sigma_Y)^2} \right] \right\}
\]
\[
= \frac{1}{\sqrt{2\pi} \sigma_Y} 2 \exp \left\{ - \frac{y^2 + (\mu_Y)^2}{2(\sigma_Y)^2} \right\} \cosh \left( \frac{y \times \mu_Y}{(\sigma_Y)^2} \right)
\]
\[
= \frac{1}{\sqrt{2\pi} \sigma_Y} 2 \cosh \left( 9n k \left( C_p \right)^2 \right) \exp \left\{ - \frac{9n \left( C_p \right)^2 k \left( y^2 + k^2 \right)} {2} \right\}
\]
\[
= 6 C_p \sqrt{\frac{n}{2\pi}} \cosh \left( 9n \left[ C_p \right]^2 k \right) \exp \left\{ - \frac{9n \left( C_p \right)^2 k \left( 1 - x \right)^2} {2} \right\}.
\]

Therefore, the probability density function $f(x)$ is
\[
f(x) = 6 C_p \sqrt{\frac{n}{2\pi}} \cosh \left\{ 9n \left[ C_p \right]^2 k \left( 1 - x \right) \right\} \exp \left\{ - \frac{9n \left( C_p \right)^2 k \left( 1 - x \right)^2} {2} \right\}.
\]
Theorem 2: The first two moments of $\hat{C}_a$ are:

$$E(\hat{C}_a) = C_a - \frac{1}{3C_p} \sqrt{\frac{2}{\pi n}} \exp \left[ \frac{-\delta}{2} \right] + 2 \left( 1 - C_a \right) \Phi \left[ -\frac{3 \sqrt{n} (C_p - C_{pk})}{\delta} \right].$$

$$E(\hat{C}_a^2) = (C_a)^2 + \frac{1}{9n (C_p)^2} - \frac{2}{3C_p} \sqrt{\frac{2}{\pi n}} \exp \left[ \frac{-\delta}{2} \right] + 4 \left( 1 - C_a \right) \Phi \left[ -\frac{3 \sqrt{n} (C_p - C_{pk})}{\delta} \right], \text{ where } \delta = 9n (C_p - C_{pk})^2.$$

Proof: For simplicity of the derivation of the exact formulae for the moments, we assume that $\mu \geq m$. For the other case, $\mu < m$, the derivation and the result will be the same. From Theorem 1, the probability density function of $\hat{k}$ is

$$g(y) = \frac{1}{\sqrt{2\pi} \sigma_r} \exp \left[ \frac{(y - \mu_r)^2}{2(\sigma_r)^2} \right] + \exp \left[ \frac{(y + \mu_r)^2}{2(\sigma_r)^2} \right].$$

Therefore,

$$E[k] = \sqrt{\frac{2}{\pi}} \sigma_r \exp \left[ - \frac{1}{2} \left( \frac{\mu_r^2}{(\sigma_r)^2} \right) \right] + \mu_r \left[ 1 - 2 \Phi \left( -\frac{\mu_r}{\sigma_r} \right) \right]$$

$$= \sqrt{\frac{2}{n \pi}} \frac{1}{3C_p} \exp \left[ - \frac{9n}{1} (C_r)^2 \right] + k \left[ 1 - 2 \Phi \left( -3 \sqrt{n} k C_p \right) \right]$$

$$= \sqrt{\frac{2}{n \pi}} \frac{1}{3C_p} \exp \left[ - \frac{9n}{2} (C_r - C_a)^2 \right] + k \left[ 1 - 2 \Phi \left( -3 \sqrt{n} k C_p \right) \right]$$

$$= \sqrt{\frac{2}{n \pi}} \frac{1}{3C_p} \exp \left[ - \frac{9n}{2} (C_r - C_{pk})^2 \right] + \left[ 1 - 2 \Phi \left( -3 \sqrt{n} (C_r - C_{pk}) \right) \right].$$

$$E(k^2) = E(Y^2) = \mu_r^2 + (\sigma_r)^2 = k^2 + \frac{1}{9n (C_p)^2}$$

$$= (1 - C_p)^2 + \frac{1}{9n (C_p)^2} = \frac{9n (C_r)^2 (1 - C_p)^2 + 1}{9n (C_p)^2} = \frac{9n (C_r - C_{pk})^2 + 1}{9n (C_p)^2}$$
= \frac{\delta + 1}{9 n (C_p)^2}, \text{ where } \delta = 9 n (C_p - C_{pk})^2.

Then, we may obtain \(E(\hat{C}_a)\) and \(E(\hat{C}_a^2)\) as:

\[
E(\hat{C}_a) = 1 - E(\hat{k})
\]

\[
= 1 - \sqrt{\frac{2}{n \pi}} \frac{1}{3 C_p} \exp \left[-\frac{\delta}{2}\right] \left[1 - 2 \Phi \left(-3 \hat{\theta} (C_p - C_{pk})\right)\right]
\]

\[
= (1 - k) \sqrt{\frac{2}{n \pi}} \frac{1}{3 C_p} \exp \left[-\frac{\delta}{2}\right] + 2 k \Phi \left(-3 \hat{\theta} (C_p - C_{pk})\right)
\]

\[
= C_a \sqrt{\frac{2}{n \pi}} \frac{1}{3 C_p} \exp \left[-\frac{\delta}{2}\right] + 2 (1 - C_a) \Phi \left(-3 \hat{\theta} (C_p - C_{pk})\right).
\]

\[
E(\hat{C}_a^2) = [E(\hat{k})]^2 - 2 E(\hat{k}) + 1
\]

\[
= k^2 + \frac{1}{9 n (C_p)^2} \left[2 \frac{1}{3 C_p} \sqrt{\frac{2}{n \pi}} \exp \left[-\frac{\delta}{2}\right] + 2 k \Phi \left(-3 \hat{\theta} (C_p - C_{pk})\right)\right] + 1
\]

\[
= (k^2 - 2 k + 1) + \frac{1}{9 n (C_p)^2} \left[2 \frac{1}{3 C_p} \sqrt{\frac{2}{n \pi}} \exp \left[-\frac{\delta}{2}\right] + 4 k \Phi \left(-3 \hat{\theta} (C_p - C_{pk})\right)\right]
\]

\[
= (C_a) + \frac{1}{9 n (C_p)^2} \left[2 \frac{1}{3 C_p} \sqrt{\frac{2}{n \pi}} \exp \left[-\frac{\delta}{2}\right] + 4 (1 - C_a) \Phi \left(-3 \hat{\theta} (C_p - C_{pk})\right)\right].
\]

**BIBLIOGRAPHY**


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