Embedding hamiltonian paths in hypercubes with a required vertex in a fixed position

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Abstract

Assume that \( n \) is a positive integer with \( n \geq 2 \). It is proved that between any two different vertices \( x \) and \( y \) of \( Q_n \) there exists a path \( P_l(x, y) \) of length \( l \) for any \( l \) with \( h(x, y) \leq l \leq 2^n - 1 \) and \( 2(l - h(x, y)) \). We expect such path \( P_l(x, y) \) can be further extended by including the vertices not in \( P_l(x, y) \) in a hamiltonian path from \( x \) to a fixed vertex \( z \) or a hamiltonian cycle. In this paper, we prove that for any two vertices \( x \) and \( z \) from different partite set of \( n \)-dimensional hypercube \( Q_n \), for any vertex \( y \in V(Q_n) - \{x, z\} \), and for any integer \( l \) with \( h(x, y) \leq l \leq 2^n - 1 - h(y, z) \) and \( 2(l - h(x, y)) \), there exists a hamiltonian path \( R(x, y, z; l) \) from \( x \) to \( z \) such that \( d_{R(x,y,z;l)}(x, y) = l \). Moreover, for any two distinct vertices \( x \) and \( y \) of \( Q_n \) and for any integer \( l \) with \( h(x, y) \leq l \leq 2^n - 1 \) and \( 2(l - h(x, y)) \), there exists a hamiltonian cycle \( S(x, y; l) \) such that \( d_{S(x,y,l)}(x, y) = l \).

Keywords: Interconnection networks

1. Introduction

In this paper, a network is represented as a loopless undirected graph. For the graph definition and notation, we follow [1]. \( G = (V, E) \) is a graph if \( V \) is a finite set and \( E \) is a subset of \( \{ (a, b) \mid (a, b) \text{ is an unordered pair of } V \} \). We say that \( V \) is the vertex set and \( E \) is the edge set. A graph \( G = (V_0 \cup V_1, E) \) is bipartite if \( V(G) \) is the union of two disjoint sets \( V_0 \) and \( V_1 \) such that every edge joins \( V_0 \) with \( V_1 \). Two vertices \( u \) and \( v \) are adjacent if \( (u, v) \in E \). A path is a sequence of adjacent vertices, written as \( \langle v_0, v_1, \ldots, v_m \rangle \), in which all the vertices \( v_0, v_1, \ldots, v_m \) are distinct except possible \( v_0 = v_m \). We also write the path \( \langle v_0, P, v_m \rangle \) where \( P = \langle v_0, v_1, \ldots, v_m \rangle \). The length of a path \( P \), denoted by \( l(P) \), is the number of edges in \( P \). Let \( u \) and \( v \) be two vertices of \( G \). The distance between \( u \) and \( v \) denoted by \( d_G(u, v) \) is the length of the shortest path of \( G \) joining \( u \) and \( v \). A cycle is a path with at least three vertices such that the first vertex is the same as the last one. A hamiltonian cycle is a cycle of length \( |V(G)| \). A hamiltonian path is a path of length \( |V(G)| - 1 \).

Interconnection networks play an important role in parallel computing/communication systems. The graph embedding problem is a central issue in evaluating a network. The graph embedding problem asked if the
quest graph is a subgraph of a host graph, and an important benefit of the graph embeddings is that we can apply existing algorithm for guest graphs to host graphs. This problem has attracted a burst of studies in recent years. Cycle networks and path networks are suitable for designing simple algorithms with low communication costs. The cycle embedding problem, which deals with all possible length of the cycles in a given graph, is investigated in a lot of interconnection networks [2, 7, 9, 11, 14]. The path embedding problem, which deals with all possible length of the paths between given two vertices in a given graph, is investigated in a lot of interconnection networks [3–6, 11, 13, 14].

Let \( u = u_n u_{n-1} \ldots u_2 u_1 \) be an \( n \)-bit binary strings. The Hamming weight of \( u \), denoted by \( w(u) \), is the number of \( i \) such that \( u_i = 1 \). Let \( u = u_n u_{n-1} \ldots u_2 u_1 \) and \( v = v_n v_{n-1} \ldots v_2 v_1 \) be two \( n \)-bit binary strings. The Hamming distance \( h(u, v) \) between two vertices \( u \) and \( v \) is the number of different bits in the corresponding strings of both vertices. The \( n \)-dimensional hypercube, denoted by \( Q_n \), consists of all \( n \)-bit binary strings as its vertices and two vertices \( u \) and \( v \) are adjacent if and only if \( h(u, v) = 1 \). Thus, \( Q_n \) is a bipartite graph with bipartition \( \{ u \mid w(u) \text{ is odd} \} \) and \( \{ u \mid w(u) \text{ is even} \} \). A vertex \( u \) of \( Q_n \) is white if \( w(u) \) is odd, otherwise \( u \) is black. It is known that \( d_{Q_n}(u, v) = h(u, v) \).

For \( i = 0, 1 \), let \( Q_{n-1}^i \) denote the subgraph of \( Q_n \) induced by \( \{ u = u_n u_{n-1} \ldots u_2 u_1 \mid u_n = i \} \). Obviously, \( Q_{n-1}^1 \) is isomorphic to \( Q_{n-1}^{\bar{1}} \). For any vertex \( u = u_n u_{n-1} \ldots u_2 u_1 \), we use \( u' \) to denote the vertex \( v = v_n v_{n-1} \ldots v_2 v_1 \) with \( u_i = v_i \) for \( 1 \leq i \leq n-1 \) and \( u_n = 1 - v_n \).

The hypercube \( Q_n \) is one of the most popular interconnection networks for parallel computer/communication system [10]. This is partly due to its attractive properties such as regularity, recursive structure, vertex and edge symmetry, maximum connectivity, as well as effective routing and broadcasting algorithm.

For the path embedding problem on hypercubes, Li et al. [11] proved that between any two different vertices \( x \) and \( y \) of \( Q_n \) there exists a path \( P_l(x, y) \) of length \( l \) for any \( l \) with \( h(x, y) \leq l \leq 2^n - 1 \) and \( 2|(l - h(x, y)) \). Note that the requirement \( 2|(l - h(x, y)) \) is needed because \( Q_n \) is a bipartite graph for every positive integer \( n \). Moreover, the requirement \( h(x, y) \leq l \leq 2^n - 1 \) is needed because the distance between \( x \) and \( y \) in \( Q_n \) is \( h(x, y) \). Obviously, we expect such path \( P_l(x, y) \) can be further extended by including the vertices not in \( P_l(x, y) \) into a hamiltonian path from \( x \) to a fixed vertex \( z \) or a hamiltonian cycle. For this reason, we prove that for any two vertices \( x \) and \( z \) from different partite set of \( Q_n \), for any vertex \( y \notin \{ x, z \} \), and for any integer \( l \) with \( h(x, y) \leq l \leq 2^n - 1 - h(y, z) \) and \( 2|(l - h(x, y)) \), there exists a hamiltonian path \( R(x, y, z, l) \) from \( x \) to \( z \) such that \( d_{R(x, y, z, l)}(x, y) = l \). As a corollary, we prove that for any \( n \geq 2 \), for any two distinct vertices \( x \) and \( y \) of \( Q_n \) and for any integer \( l \) with \( h(x, y) \leq l \leq 2^n - 1 \) and \( 2|(l - h(x, y)) \) there exists a hamiltonian cycle \( S(x, y; l) \) such that \( d_{S(x, y; l)}(x, y) = l \).

In the following section, we introduce another interesting property, called 2RP-property, of hypercubes. We defer the proof of 2RP-property for hypercube in Section 3. Instead we prove that many interesting properties, including the aforementioned properties, of hypercube is a direct consequence of 2PR-property. In Section 4, we give a discussion on 2RP-property.

### 2. 2RP-property

Assume that \( n \) is any positive integer with \( n \geq 2 \). Let \( u \) and \( x \) be two distinct white vertices of \( Q_n \) and \( y \) and \( z \) be two distinct black vertices of \( Q_n \). It is proved in [8] that there are two disjoint paths \( P_1 \) and \( P_2 \) such that \( (1) \ P_1 \) is a path joining \( u \) to \( v \), \( (2) \ P_2 \) is a path joining \( x \) to \( y \), and \( (3) \ P_1 \cup P_2 \) spans \( Q_n \). We call such property the 2P property. The 2P property has been used in many applications of hypercubes [8, 12]. Obviously, the lengths of \( P_1 \) and \( P_2 \) satisfy \( l(P_1) + l(P_2) = 2^n - 2 \). Yet, we can further require that the length of \( P_1 \), and hence the length of \( P_2 \), can be any odd integer such that \( l(P_1) \geq h(u, v) \) and \( l(P_2) \geq h(x, y) \). We call such property to be the 2RP property. More precisely, let \( u \) and \( x \) be two distinct white vertices of \( Q_n \) and \( v \) and \( y \) be two distinct black vertices of \( Q_n \). Let \( l_1 \) and \( l_2 \) are odd integers with \( l_1 \geq h(u, v) \), \( l_2 \geq h(x, y) \), and \( l_1 + l_2 = 2^n - 2 \). Then there are two disjoint paths \( P_1 \) and \( P_2 \) such that \( (1) \ P_1 \) is a path joining \( u \) to \( v \) with \( l(P_1) = l_1 \), \( (2) \ P_2 \) is a path joining \( x \) to \( y \) with \( l(P_2) = l_2 \), and \( (3) \ P_1 \cup P_2 \) spans \( Q_n \). In next section, we will formally prove the following theorem.

**Theorem 2.1.** \( Q_n \) satisfies the 2RP property if \( n \geq 4 \).

Now, we make some remarks to illustrate that some interesting properties of hypercubes are consequences of Theorem 2.1.

**Remark 1.** The hamiltonian laceable property of hypercubes, proved in [15], states that there exists a hamiltonian path of \( Q_n \) joining any white vertex \( u \) to any black vertex \( y \). Now, we prove that \( Q_n \) is hamiltonian laceable by Theorem 2.1. Obviously, \( Q_n \) is hamiltonian laceable for \( n = 1, 2, 3 \). Since \( n \geq 4 \), we can choose a pair of adjacent vertices \( v \) and \( x \) such that \( v \) is a black vertex with \( v \neq y \) and \( x \) be a white vertex with \( x \neq u \).
By Theorem 2.1, there are two disjoint paths $P_1$ and $P_2$ such that (1) $P_1$ is a path joining $u$ to $v$ with $l(P_1) = 2^n - l - 1$, (2) $P_2$ is a path joining $u$ to $v$ with $l(P_2) = l - 1$, and (3) $P_1 \cup P_2$ spans $Q_n$. Obviously, $(u, P_1, v, P_2, y)$ forms a hamiltonian path joining $u$ to $y$. Thus, $Q_n$ is hamiltonian laceable.

**Remark 2.** The bipanconnected property of $Q_n$, proved in [11], stated that between any two different vertices $x$ and $y$ of $Q_n$, there exists a path $P(x, y)$ of length $l$ for any $l$ with $h(x, y) \leq l \leq 2^n - 1$ and $2(l - h(x, y))$. Now, we prove that $Q_n$ is bipanconnected by Theorem 2.1. Obviously, $Q_n$ is bipanconnected for $n = 1, 2, 3$. Now, we consider $n \geq 4$. Without loss of generality, we assume that $x$ is a white vertex.

Suppose that $y$ is a black vertex. Thus, $h(x, y)$ is odd. Let $l$ be any odd integer with $h(x, y) \leq l \leq 2^n - 1$. Suppose that $l = 2^n - 1$. By Remark 1, $Q_n$ is hamiltonian laceable. Obviously, the hamiltonian path of $Q_n$ joining $x$ and $y$ is of length $2^n - 1$. Suppose that $l < 2^n - 1$. Since $n \geq 4$, we can choose a pair of adjacent vertices $u$ and $v$ such that $u$ is a white vertex with $u \neq x$ and $v$ be a black vertex with $v \neq y$. Obviously, $h(u, v) = 1$. By Theorem 2.1, there exist two disjoint paths $P_1$ and $P_2$ such that (1) $P_1$ is a path joining $u$ to $v$ with $l(P_1) = 2^n - 2 - l$, (2) $P_2$ is a path joining $x$ to $y$ with $l(P_2) = l$, and (3) $P_1 \cup P_2$ spans $Q_n$. Obviously, $P_2$ is a path of length $l$ joining $x$ to $y$.

Suppose that $y$ is a white vertex. Thus, $h(x, y)$ is even. Let $l$ be any even integer with $h(x, y) \leq l < 2^n - 1$. Since $n \geq 4$, we can choose two different neighbors $u$ and $v$ of $y$ such that $h(u, v) = h(x, y) - 1$. By Theorem 2.1, there exist two disjoint paths $P_1$ and $P_2$ such that (1) $P_1$ is a path joining $x$ to $u$ with $l(P_1) = l - 1$, (2) $P_2$ is a path joining $y$ to $v$ with $l(P_2) = 2^n - l - 1$, and (3) $P_1 \cup P_2$ spans $Q_n$. Obviously, $(x, P_1, u, y)$ is a path of length $l$ joining $x$ to $y$.

Thus, $Q_n$ is bipanconnected.

**Remark 3.** The edge-bipancyclic property of $Q_n$, proved in [11], stated that for any edge $e = (x, y)$ and for any even integer with $4 \leq l \leq 2^n$ there exists a cycle of length $l$ containing the edge $e$ if $n \geq 2$. Again, we prove that $Q_n$ is edge-bipancyclic by Theorem 2.1. Obviously, $Q_n$ is edge-bipancyclic for $n = 2, 3$. Thus, we consider $n \geq 4$. Suppose that $l = 2^n$. By Remark 1, there exists a hamiltonian path $P$ joining $x$ to $y$. Obviously, $(x, P, y, x)$ forms a hamiltonian cycle of length $2^n$ containing the edge $e$. Suppose that $l < 2^n$. Since $n \geq 4$, we can choose a pair of adjacent vertices $u$ and $v$ such that $u$ is a white vertex. By Theorem 2.1, there exist two disjoint paths $P_1$ and $P_2$ such that (1) $P_1$ is a path joining $u$ to $v$ with $l(P_1) = 2^n - l - 1$, (2) $P_2$ is a path joining $x$ to $y$ with $l(P_2) = l - 1$, and (3) $P_1 \cup P_2$ spans $Q_n$. Obviously, $(x, P_2, y, x)$ is a cycle of length $l$ containing the edge $e$. Thus, $Q_n$ is edge-bipancyclic for $n \geq 2$.

The following results are also consequence of Theorem 2.1.

**Theorem 2.2.** Assume that $n$ is any positive integer with $n \geq 2$. Let $x$ and $z$ be two vertices from different partite set of $Q_n$ and $y$ be a vertex of $Q_n$ that is not in $[x, z]$. For any integer $l$ with $h(x, y) \leq l \leq 2^n - 1 - h(y, z)$ and $2(l - h(x, y))$, there exists a hamiltonian path $R(x, y; z, l)$ from $x$ to $z$ such that $d_R(x, y; z, l)(x, y) = l$.

**Proof.** By brute force, we can check the theorem holds for $n = 2, 3$. Now, we consider $n \geq 4$. Without loss of generality, we assume that $x$ is a white vertex and $z$ is a black vertex.

Suppose that $y$ is a black vertex. Obviously, $h(y, z) \geq 2$. There exists a neighbor $w$ of $y$ such that $w \neq x$ and $h(w, z) = h(y, z) - 1$. Obviously, $w$ is a white vertex. By Theorem 2.1, there exist two disjoint paths $R_1$ and $R_2$ such that (1) $R_1$ is a path joining $x$ to $w$ with $l(R_1) = l$, (2) $R_2$ is a path joining $w$ to $z$ with $l(R_2) = 2^n - l - 2$, and (3) $R_1 \cup R_2$ spans $Q_n$. We set $R$ as $(x, R_1, y, w, R_2, z)$. Obviously, $R$ is the required hamiltonian path.

Suppose that $y$ is a white vertex. Obviously, $h(x, y) \geq 2$. There exists a neighbor $w$ of $y$ such that $w \neq z$ and $h(w, x) = h(y, x) - 1$. Obviously, $w$ is a black vertex. By Theorem 2.1, there exist two disjoint paths $R_1$ and $R_2$ such that (1) $R_1$ is a path joining $x$ to $w$ with $l(R_1) = l - 1$, (2) $R_2$ is a path joining $y$ to $z$ with $l(R_2) = 2^n - l - 1$, and (3) $R_1 \cup R_2$ spans $Q_n$. We set $R$ as $(x, R_1, w, y, R_2, z)$. Obviously, $R$ is the required hamiltonian path.

**Corollary 2.1.** Assume that $n$ is any positive integer with $n \geq 2$. Let $x$ and $y$ be any two different vertices of $Q_n$. For any integer $l$ with $h(x, y) \leq l \leq 2^{n-1}$ there exists a hamiltonian cycle $S(x, y; l)$ such that $d_S(x, y; l)(x, y) = l$ and $2(l - h(x, y))$.

**Proof.** Let $z$ be a neighbor of $x$ such that $z \neq y$. By Theorem 2.2, there exists a hamiltonian path $R$ joining $x$ to $z$ such that $d_R(x, y; z, l)(x, y) = l$. We set $S$ as $(x, R, z, x)$. Obviously, $S$ forms the required hamiltonian cycle.
3. Proof of Theorem 2.1

Now, we prove Theorem 2.1. By brute force, we can check the theorem holds for any $Q_k$ with $4 \leq k < n$. Without loss of generality, we can assume that $l_1 \geq l_2$. Thus, $l_2 \leq 2^{n-1} - 1$. Since $Q_n$ is edge symmetric, we can assume that $u \in V(Q^0_n-1)$ and $x \in V(Q^1_n-1)$. We have the following cases.

**Case 1.** $v \in V(Q^0_n-1)$ and $y \in V(Q^1_n-1)$. Suppose that $l_2 < 2^{n-1} - 1$. By Remark 1, there exists a hamiltonian path $r$ of $Q^0_n-1$ joining $u$ and $v$. Since the length of $R$ is $2^{n-1} - 1$, we can write $R$ as $(u, R_1, p, q, R_2, v)$ for some black vertex $p$ with $p^n \neq x$ and some white vertex $q$ with $q^n \neq y$. Obviously, $h(p^n, q^n) = 1$. By induction, there exist two disjoint paths $S_1$ and $S_2$ such that (1) $S_1$ is a path joining $p^n$ to $q^n$ with $l(S_1) = l_1 - 2^{n-1}$, (2) $S_2$ is a path joining $x$ to $y$ with $l(S_2) = 2^{n-1} - 3$, and (3) $S_1 \cup S_2$ spans $Q^1_n-1$. We set $P_1$ as $(u, R_1, p, p^n, q, S_2, v)$ and set $P_2$ as $S_2$. Obviously, $P_1$ and $P_2$ are the required paths. See Fig. 1(a) for illustration.

Suppose that $l_2 = 2^{n-1} - 1$. By Remark 1, there exists a hamiltonian path $P_1$ of $Q^0_n-1$ joining $u$ and $v$ and there exists a hamiltonian path $P_2$ of $Q^0_n-1$ joining $x$ and $y$. Obviously, $P_1$ and $P_2$ are the required paths. See Fig. 1(b) for illustration.

**Case 2.** $(v, y) \subset V(Q^1_n-1)$. Suppose that $l_2 < 2^{n-1} - 1$. We choose a neighbor $p$ of $v$ such that $p \neq x$. Obviously, $p$ is a white vertex. By induction, there exist two disjoint paths $S_1$ and $S_2$ such that (1) $S_1$ is a path joining $p$ to $v$ with $l(S_1) = l_1 - 2^{n-1}$, (2) $S_2$ is a path joining $x$ to $y$ with $l(S_2) = l_2$, and (3) $S_1 \cup S_2$ spans $Q^1_n-1$. By Remark 1, there exists a hamiltonian path $R$ of $Q^0_n-1$ joining $u$ and $p^n$. We set $P_1$ as $(u, R, p^n, p, S_1, v)$ and we set $P_2$ as $S_2$. Obviously, $P_1$ and $P_2$ are the required paths. See Fig. 1(c) for illustration.

Suppose that $l_2 = 2^{n-1} - 1$. Again, we choose a neighbor $p$ of $v$ such that $p \neq x$. By induction, there exist two disjoint paths $S_1$ and $S_2$ such that (1) $S_1$ is a path joining $p$ to $v$ with $l(S_1) = l_1 - 2^{n-1}$, (2) $S_2$ is a path joining $x$ to $y$ with $l(S_2) = 2^{n-1} - 3$, and (3) $S_1 \cup S_2$ spans $Q^1_n-1$. Obviously, we can write $S_2$ as $(x, S^n_2, r, s, S^n_2, y)$ for some black vertex $r$ with $r^n \neq u$. Again by induction, there exist two disjoint paths $R_1$ and $R_2$ such that (1) $R_1$ is a path joining $u$ to $p^n$ with $l(R_1) = 2^{n-1} - 3$, (2) $R_2$ is a path joining $r^n$ to $s^n$ with $l(R_2) = 1$, and (3) $R_1 \cup R_2$ spans $Q^0_n-1$. We set $P_1$ as $(u, R_1, p^n, p, v)$ and set $P_2$ as $(x, S^n_2, r, r^n, s^n, s, S^n_2, y)$. Obviously, $P_1$ and $P_2$ are the required paths. See Fig. 1(d) for illustration.

**Case 3.** $y \in V(Q^0_n-1)$ and $v \in V(Q^1_n-1)$. Suppose that $l_2 = 1$. Obviously, $x = y^n$. Let $p$ be a neighbor of $y$ in $Q^0_n-1$ such that $y^n \neq v$ and let $q$ be a neighbor of $p$ in $Q^0_n-1$ such that $p \neq y$. By induction, there exist two disjoint paths $R_1$ and $R_2$ such that (1) $R_1$ is a path joining $u$ to $q$ and $l(R_1) = 2^{n-1} - 3$, (2) $R_2$ is a path joining $p$ to $y$ and $l(R_2) = 1$, and (3) $R_1 \cup R_2$ spans $Q^0_n-1$. Obviously, $p^n$ is a black vertex and $q^n$ is a white vertex. Again by induction, there exist two disjoint paths $S_1$ and $S_2$ such that (1) $S_1$ is a path joining $q^n$ to $v$ with $l(S_1) = 2^{n-1} - 3$, (2) $S_2$ is a path joining $x$ to $p^n$ with $l(S_2) = 1$, and (3) $S_1 \cup S_2$ spans $Q^1_n-1$. We set $P_1$ as $(u, R_1, q, p, p^n, q^n, S_1, v)$ and set $P_2$ as $(x, y)$. Obvi-

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**Fig. 1. Illustration for Theorem 2.1.**
ously, $P_1$ and $P_2$ are the required paths. See Fig. 1(e) for illustration.

Suppose that $l_2 \geq 3$. We set $p$ be a neighbor in $Q_{n-1}^0$ of $y$ with $p \neq u$ if $h(x, y) = 1$ and set $q$ be a neighbor of $y$ in $Q_{n-1}^0$ with $p \neq u$ and $h(p, y) = h(x, y) - 1$ if $h(x, y) = 3$. Let $q$ be a neighbor $v^n$ in $Q_{n-1}^0$ such that $q \neq y$ and $q' \neq x$. Thus, $h(q^n, v) = 1$. By induction, there exist two disjoint paths $R_1$ and $R_2$ such that $(1) R_1$ is a path joining $u$ to $p$ with $l(R_1) = 2n - 3$, $(2) R_2$ is a path joining $q$ to $y$ with $l(R_2) = 1$, and $(3) R_1 \cup R_2$ spans $Q_{n-1}^0$. Again by induction, there exist two disjoint paths $S_1$ and $S_2$ such that $(1) S_1$ is a path joining $q^n$ to $v$ with $l(S_1) = l_1 - 2n + 2$, $(2) S_2$ is a path joining $x$ to $p^n$ with $l(S_2) = l_2 - 2$, and $(3) S_1 \cup S_2$ spans $Q_{n-1}^0$. We set $P_1$ as $(u, R_1, p, q^n, S_1, v)$ and set $P_2$ as $(x, S_2, p^n, q, R_2, v)$. Obviously, $P_1$ and $P_2$ are the required paths. See Fig. 1(f) for illustration.

**Case 4.** $(v, y) \in V(Q_{n-1}^0)$. Suppose that $l_2 = 1$. Obviously, $y = x^0$. By Remark 1, there exist a hamiltonian path $R$ of $Q_{n-1}^0$ joining $u$ to $v$. Obviously, $R$ can be written as $(u, R_1, p, y, q, R_2, v)$. Note that $u = p$ if $l(R_1) = 0$. Obviously, $p$ and $q$ are white vertices. Thus, $p^n$ and $q^n$ are black vertices. Let $r$ be a neighbor of $q^n$ in $Q_{n-1}^0$ such that $r \neq x$. By induction, there exist two disjoint paths $S_1$ and $S_2$ such that $(1) S_1$ is a path joining $p^n$ to $r$ with $l(S_1) = 2n - 3$, $(2) S_2$ is a path joining $q^n$ to $x$ with $l(S_2) = 1$, and $(3) S_1 \cup S_2$ spans $Q_{n-1}^0$. We set $P_1$ as $(u, R_1, p, q^n, S_1, r, q^n, q, R_2, v)$ and set $P_2$ as $(x, S_2, p^n, y, R_2, v)$. Obviously, $P_1$ and $P_2$ are the required paths. See Fig. 1(g) for illustration.

Suppose that $l_2 \geq 3$. We set $p$ be a neighbor of $y$ in $Q_{n-1}^0$ with $p \neq u$ if $h(x, y) = 1$ and set $q$ be a neighbor of $y$ in $Q_{n-1}^0$ with $p \neq u$ and $h(p, y) = h(x, y) - 1$ if $h(x, y) = 3$. By induction, there exist two disjoint paths $R_1$ and $R_2$ such that $(1) R_1$ is a path joining $u$ to $v$ with $l(R_1) = 2n - 3$, $(2) R_2$ is a path joining $p$ to $y$ with $l(R_2) = 1$, and $(3) R_1 \cup R_2$ spans $Q_{n-1}^0$. Obviously, we can write $R_1$ as $(u, R_1, p, s, t, R_2, v)$ for some black vertex $s$ such that $s \neq x$. By induction, there exist two disjoint paths $S_1$ and $S_2$ such that $(1) S_1$ is a path joining $s^n$ to $t^n$ with $l(S_1) = l_1 - 2n + 2$, $(2) S_2$ is a path joining $x$ to $p^n$ with $l(S_2) = l_2 - 2$, and $(3) S_1 \cup S_2$ spans $Q_{n-1}^0$. We set $P_1$ as $(u, R_1, s, s^n, t, t^n, R_2, v)$ and set $P_2$ as $(x, S_2, p^n, q, R_2, v)$. Obviously, $P_1$ and $P_2$ are the required paths. See Fig. 1(h) for illustration.

4. Discussion

Since there are four vertices in $Q_2$, it is easy to check that $Q_2$ satisfies the 2RP-property. However, the 2RP-property does not hold for $Q_3$. Let $u = 000$, $v = 111$, $x = 011$, and $y = 001$. By brute force, we can check that we cannot find two disjoint paths $P_1$ and $P_2$ such that $P_1$ is a path joining $u$ to $v$ with $l(P_1) = 3$ and $P_2$ is a path joining $x$ to $y$ with $l(P_2) = 3$.

By changing the roles of the vertices in bipartite sets in Theorem 2.1, we have the following theorem. The proof is similar to the proof of Theorem 2.1.

**Theorem 4.1.** Assume that $n$ is a positive integer with $n \geq 4$. Let $u$ and $v$ be two distinct white vertices of $Q_n$ and $x$ and $y$ be two distinct black vertices of $Q_n$. Let $l_1$ and $l_2$ be even integers with $l_1 \geq h(u, x)$, $l_2 \geq h(v, y)$, and $l_1 + l_2 = 2n - 2$ except for the case $\{l_1, l_2\} = \{2, 2n - 4\}$ with $\{u, x, v, y\}$ inducing a cycle of length 4. There exist two disjoint paths $P_1$ and $P_2$ such that $(1) P_1$ is a path joining $u$ to $x$ and $l(P_1) = l_1$, $(2) P_2$ is a path joining $v$ to $y$ and $l(P_2) = l_2$, and $(3) P_1 \cup P_2$ spans $Q_n$.

Suppose that $n = 3$. Let $u = 000$, $v = 011$, $x = 100$, and $y = 111$. We can check that there are no two disjoint paths $P_1$ and $P_2$ such that $P_1$ is a path joining $u$ to $v$ and $P_2$ is a path joining $x$ to $y$ such that $P_1 \cup P_2$ spans $Q_3$. Again, the above theorem does not hold for $n = 3$.

References


